The Jerk Dynamics of Lorenz Model

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1 Introduction

At the very beginning of the 1960s, Edward Norton Lorenz (1917–2008), a meteorologist from the famous MIT (Massachusetts Institute of Technology), succeeded in establishing a model for atmospheric convection comprising only three variables. The solution of this weather forecasting model that Lorenz [13] plotted in a three-dimensional phase space is compelled to evolve on a chaotic attractor which resembles the wings of a butterfly. It is probably this form that prompted Lorenz to call the "sensitivity to initial conditions" (described by the French mathematician Henri Poincaré as early as 1908 in his philosophical writings *Science and Method* [14]) the "butterfly effect." During these last two decades, the seminal works of Gottlieb [5] and Sprott [18–24] have triggered out an increasing interest in the study of chaotic oscillators based on jerk equations, that is, oscillators which can be completely described by third-order ordinary differential equations of the form $\dddot{x} = f(\ddot{x}, \dot{x}, x)$. In 1997, Stephan J. Linz [11] proposed in a very interesting paper an "exact transformation" enabling to obtain the jerk form of the Lorenz model and a nonlinear transformation "simplifying its jerky dynamics." Unfortunately, the third-order nonlinear differential equation he finally obtained precluded any

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mathematical analysis and made difficult numerical investigations since it contained exponential functions. Let's notice that the jerk form in x of the Lorenz model that we will provide below is exactly the same as those obtained by Linz but presented in a different way. In 2014, Buscarino et al. [2] used *linear combinations* of the three nonlinear ordinary differential equations modeling Chua's circuit to deduce its jerk forms in x and z. Recently, Xu and Cao [26] proposed to use the so-called controllable canonical form to provide all the jerk form dynamics of Chua's circuit. In this paper, following the method of *linear combinations* proposed by Buscarino et al. [2], we provide the jerk form in x of Lorenz model. Thus, by making a comparison of fixed points and their stability, eigenvalues, Lyapunov characteristic exponents (LCEs), and attractor shapes between the original three-order Lorenz model and its first jerk form in x, we demonstrate the *topological equivalence* of both systems.

The paper is organized as follows. In the next section, some introductive and historical materials on jerk equations are reported. Then, the set of three first-order nonlinear differential equations of Lorenz model and mathematical and numerical results concerning its main features (fixed points and their stability, Hopf bifurcation parameter, Lyapunov characteristic exponents, and Kaplan-Yorke dimension) are recalled. The mathematical demonstration leading to the Lorenz jerk system is then presented. A comparison with the original Lorenz model enables to show that they have of course exactly the same features. Phase portraits are also *topologically equivalent*, the only difference being the range of values of the y and z variables which is increased respectively by a factor 5 for y and 50 for z.

2 Preliminaries

At the end of the 1970s, Stephen H. Schot [15] published a very interesting paper in which he made a short history of the origin of the use of the concept of jerk, i.e., the time rate of change of the acceleration. He explained that:

> The French geometer Transon in 1845 was probably the first to consider the third time derivative of distance in mechanics and he uses the term virtualité for it. Transon computes the normal component of the jerk and expresses it in terms of what is now called the aberrancy (he uses the term déviation de courbure). Other early writers who treat the third derivative use neutral terms such as second acceleration or higher acceleration for it. Thus, Resal resolves the jerk along a space curve into tangential, normal, and binormal components and Somoff first establishes recursion formulas for these components of the higher-order accelerations in terms of those of the ordinary acceleration. Subsequently, the term jerk for the second acceleration seems to have gradually entered the literature of physics and without any explicit rational explanation for its use.

According to Pr. Christian Mira (personal communication), at the end of the nineteenth century, the French scientist Martin Haag (not to be confused with Jules Haag) made use of the concept of "over-acceleration" or jerk in a mathematical analysis [6]. Then, as recalled by Schot [15]:

> An unequivocal definition of the jerk as "the derivative of acceleration with respect to time" was given in 1928 by Melchior who justifies the use of the term by referring to the physiological sensation experienced by large changes in acceleration ... The term is now commonly used in mechanics and is being adopted in other areas of physics as well.

More than half a century later, Julian Clinton Sprott [17] found probably the simplest three-dimensional first-order system for chaos. Two years after, Hans Gottlieb [5] wondered "what is the simplest jerk function that gives chaos?", where the jerk function is of the form $\dddot{x} = f(x, \dot{x}, \ddot{x})$. According to Buscarino et al. [2]: "If a mechanical interpretation of the variable x in terms of displacement is given, the jerk equation can be viewed as an equation where the derivative of the acceleration is involved, that is, where a measure of the instantaneous variation of the acceleration is included." Since the question of the simplest jerk function is still open, in this work, we will only focus on the *topological equivalence* between the original model and its corresponding jerk equations. To this aim, we will use the Lyapunov characteristic exponents, the bifurcation diagram, and the shape of the attractor in the phase space to show that both laser's minimal universal model and its jerk equations in z are topologically equivalent. In general, given a nonlinear system of order n and considered one of its n state variables, say variable x_i, there is no guarantee that it can be rewritten in the equivalent form:

$$\frac{d^n x_i}{dt^n} = f\left(\frac{d^{(n-1)} x_i}{dt^{n-1}}, \frac{d^{(n-2)} x_i}{dt^{n-2}}, \ldots, x_i\right). \tag{1}$$

Although Xu and Cao [26] have proposed a scheme to implement the jerk form of the Chua system family using a controllable canonical form applied in linear systems, it does not seem (to our knowledge) that this scheme has been used for more general system such as laser's minimal universal model. So, we will follow in this work the method used by Buscarino et al. [2]. Thus, when it is possible to derive a jerk equation, the state space normal form of the system can be also written as follows:

$$\begin{aligned}
\frac{d\tilde{x}_1}{dt} &= \tilde{x}_2, \\
\frac{d\tilde{x}_2}{dt} &= \tilde{x}_3, \\
&\vdots \\
\frac{d\tilde{x}_n}{dt} &= f\left(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}\right),
\end{aligned} \tag{2}$$

with $\tilde{x}_1 = x_i$.

3 Lorenz Model

The purpose of the model established by Edward Lorenz [13] was in the beginning to analyze the unpredictable behavior of weather. After having developed nonlinear partial derivative equations starting from the thermal equation and Navier-Stokes equations, Lorenz truncated them to retain only three modes. The most widespread form of the Lorenz model is as follows:

$$\begin{aligned} \frac{dx}{dt} &= \sigma\left(y - x\right), \\ \frac{dy}{dt} &= -xz + rx - y, \\ \frac{dz}{dt} &= xy - \beta z, \end{aligned} \tag{3}$$

where parameters represent respectively the Prandtl number (σ), the Rayleigh number (r), and the aspect ratio of the convection cylinders (β). In this study, we will use the following parameter values: $(\sigma, r, \beta) = (10, 28, 8/3)$. With this parameter set, numerical integration of Lorenz model (3) has led to the famous strange attractor in the shape of a butterfly (see Figs. 1a,c). Dynamics features of the Lorenz model have been completely analyzed for many years in many works, the most famous of which being that of Sparrow [16]. Lorenz model (3) has three *fixed points*:

$$\begin{aligned} &O\left(0,0,0\right), \quad I\left(-\sqrt{\beta(r-1)}, -\sqrt{\beta(r-1)}, r-1\right), \\ &J\left(\sqrt{\beta(r-1)}, \sqrt{\beta(r-1)}, r-1\right), \end{aligned} \tag{4}$$

With this parameter set, eigenvalues corresponding to each of these fixed points are the following:

$$\begin{aligned} &\left(-22.8277, -8/3, 11.8277\right), \\ &\left(-13.8546, 0.09395 - 10.1945, 0.09395 + 10.1945i\right), \\ &\left(-13.8546, 0.09395 - 10.1945, 0.09395 + 10.1945i\right). \end{aligned} \tag{5}$$

Thus, the origin O is a *saddle-node*, while I and J are *saddle-foci*. Then, according to Sparrow [16], a Hopf bifurcation [1, 7, 10, 12] occurs when the parameter r reaches the value:

$$r_H = \sigma \frac{\sigma + \beta + 3}{\sigma - \beta - 1} \tag{6}$$

With the original parameter set, i.e., with $\sigma = 10$ and $\beta = 8/3$, Sparrow [16] finds: $r_H = 470/19 \approx 24.74$. In order to complete the analysis of the effects of the control parameter, value r changes on the topology of the attractor of the Lorenz model (3); the bifurcation diagram has been built and plotted in Fig. 2a for $r \in [20, 80]$. Then, by using the Lyapunov Exponents Toolbox (LET) developed by Steve Siu for MatLab® and involving the two algorithms proposed by Wolf et al. [25] and Eckmann and Ruelle [3] (see https://fr.mathworks.com/matlabcentral/fileexchange/233-let), we have obtained for this parameter set the following Lyapunov characteristic exponents (LCEs) for the Lorenz model:

$$(+0.906, 0, -14.572) \tag{7}$$

The Kaplan-Yorke conjecture [8] enabling to estimate the *fractal dimension* of a strange attractor is then equal to $d_{KY} \approx 2.062$. Thus, according to the classification of Klein and Baier [9] for (autonomous) continuous-time attractors of dynamical system, such LCEs (7) confirm the chaotic feature of the so-called Lorenz butterfly.

4 Lorenz Jerk System

Starting from the first equation of Lorenz model (3), we obtain:

$$y = \frac{\dot{x}}{\sigma} + x. \tag{8}$$

It follows that:

$$\dot{y} = \frac{\ddot{x}}{\sigma} + \dot{x}. \tag{9}$$

From the second equation of (3), we deduce that:

$$z = r - \frac{y + \dot{y}}{x}. \tag{10}$$

By replacing in this Eq. (10), y and $\dot{y}$ by their above expressions Eqs. (8 and 9), we have:

$$z = r - 1 - \frac{1}{\sigma x}\left[\ddot{x} + (\sigma + 1)\,\dot{x}\right]. \tag{11}$$

By taking the time derivative of Eq. (11), we find:

$$\dot{z} = \frac{\dot{x}}{\sigma x^2}\left[\ddot{x} + (\sigma + 1)\,\dot{x}\right] - \frac{1}{\sigma x}\left[\dddot{x} + (\sigma + 1)\,\ddot{x}\right]. \tag{12}$$

Then, by replacing equation (12) in the third equation of (3), we obtain finally:

$$\dddot{x} = -(\sigma+1)\,\ddot{x} + [\ddot{x} + (\sigma+1)\,\dot{x}]\,\frac{\dot{x}}{x} + \beta\,(r-1)\,\sigma x \\ -\beta\,[\ddot{x} + (\sigma+1)\,\dot{x}] - x^2\,(\dot{x} + \sigma x)\,. \tag{13}$$

Then, the jerk form in x is obtained by posing:

$$\dot{x} = y, \quad \ddot{x} = z, \quad \dddot{x} = f\,(x, \dot{x}, \ddot{x})\,. \tag{14}$$

Considering the Lorenz model (3), we obtain the dynamics of the jerk system:

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= z, \\ \frac{dz}{dt} &= \beta\,(r-1)\,\sigma x - \beta\,(\sigma+1)\,y - (\beta+\sigma+1)\,z \\ &\quad + [z + (\sigma+1)\,y]\,\frac{y}{x} - x^2\,(\sigma x + y)\,, \end{aligned} \tag{15}$$

Remark 1 Let's notice that Eq. (13) is identical to that obtained by Linz [11] (see his equation (18)), except for the fact that he has posed for unknown reasons $\dot{x}/x = \ln x$ and used a "Cole-Hopf-like transformation" to express his jerk equation. Thus, his resulting third-order nonlinear differential equation (20) contains exponential functions which preclude any mathematical analysis and made difficult numerical investigations.

Equations (15) represent in different state space representations Lorenz model (3) and thus maintain its same structural properties. The three-dimensional attractors for the previously defined parameters and for (σ, r, β) = (10, 28, 8/3) are reported in the following figures. The original Lorenz model chaotic attractor is reported in Figs. 1a,c, and the attractor of the equivalent jerk system represented by Eqs. (15) is reported in Figs. 1b,d.

In order to state the topological equivalence between the original representation of the Lorenz model (3) and its jerk form in x (15), we have performed a stability analysis including the fixed points stability, the occurrence of Hopf bifurcation, the representation of the bifurcation diagram, and the computation of the Lyapunov characteristic exponents for the jerk form in x (15) that we have compared to the stability analysis of the Lorenz model (3).

By using the classical nullclines method, it can be shown that the Lorenz jerk system (15) admits exactly the same fixed points (4) as the Lorenz model (3). With this parameter set, we have verified that both eigenvalues corresponding to the fixed points I and J are exactly the same as the Lorenz model (3) but are different for the origin O which is a saddle-focus $(5.99248, -9.32958 \pm 5.57737i)$ for the Lorenz jerk system (15). We have also verified that a Hopf bifurcation occurs for the same

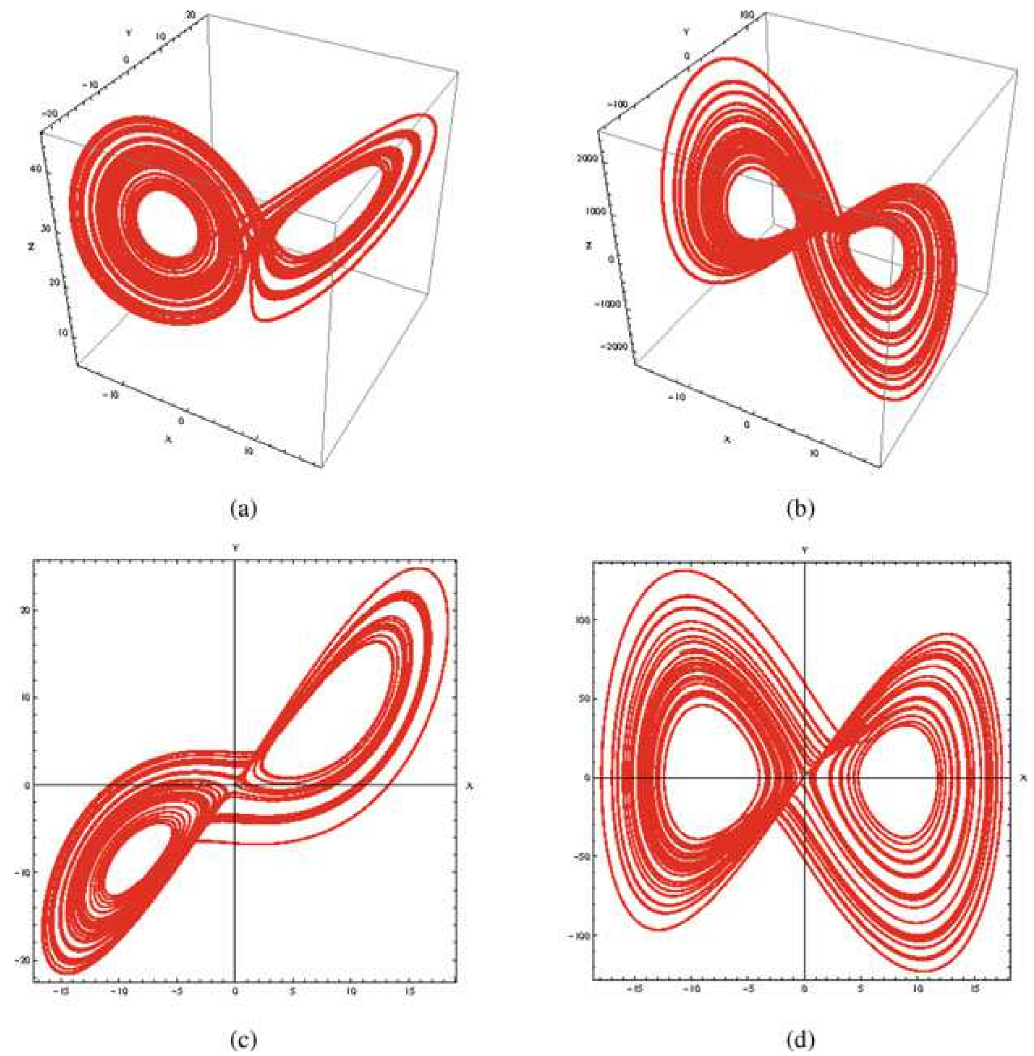


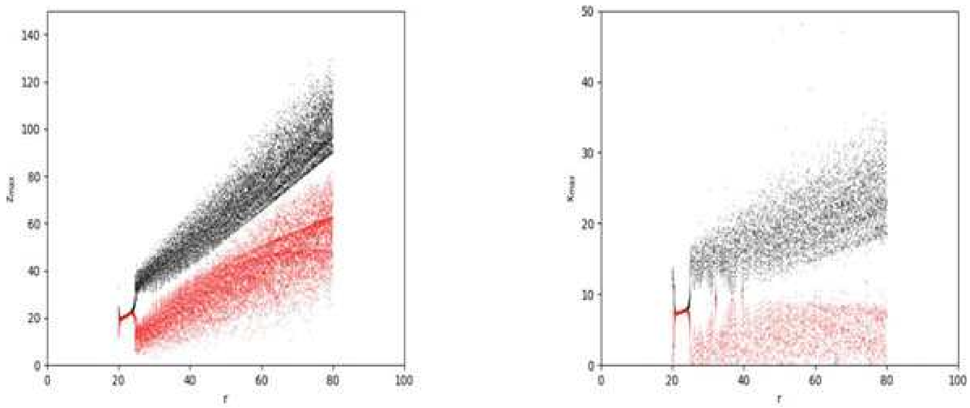
Fig. 1 Phase portraits of the Lorenz model (3) (left) and of its jerk form in x (15) (right)

value of parameter r (6). Then, in the next two Figs. 2a,b, we have plotted both bifurcation diagrams of the Lorenz model (3) and its corresponding jerk form (15). Figures 2a,b clearly demonstrate the equivalence of the two representations.

Finally, still using the Lyapunov Exponents Toolbox (LET) developed by Steve Siu for MatLab®, we have obtained for this parameter set exactly the same Lyapunov characteristic exponents (LCEs) as for the Lorenz model (7) and, of course, the same Kaplan-Yorke *fractal dimension* for the strange attractor of the Lorenz jerk system (15).

5 Discussion

In this paper, the jerk form in x of the Lorenz model has been derived following the method of *linear combinations* used by Buscarino et al. [2]. Then, a stability



(a) (b)

Fig. 2 Bifurcation diagrams of the Lorenz model (3) (left) and of its jerk form in x (15) (right)

analysis of the jerk dynamics of Lorenz model proves that fixed points and their stability, eigenvalues, Lyapunov characteristic exponents, and of course attractor shape are exactly the same as those of the original Lorenz model, the only difference being the range of values of the y and z variables which is increased respectively by a factor 5 for y and 50 for z as also highlighted on the above bifurcation diagrams (see Fig. 2a,b). Recently, Xu and Cao [26] proposed to use the so-called controllable canonical form to provide all the jerk form dynamics of Chua's circuit. So, two perspectives could be given to this work. The first would be to verify if the jerk form in x of the Lorenz model can be also obtained by making use of the controllable canonical form. The second would be an electronic realization of the jerk dynamics of Lorenz model. Concerning implications and applications of such results, let's recall that in a previous publication [4], by using the *flow curvature method*, we have been able to state a link between the *curvature* and *energy* of planar generalized Liénard systems. This has been made possible because this kind of system of two ordinary differential equations of order one can be transformed into a nonlinear second-order ordinary differential equation. The transformation of systems of three ordinary differential equations of order one into a third-order ordinary differential equation, i.e., a jerk equation, thus enables to open some promising developments for assessing the energy of nonlinear and chaotic three-dimensional dynamical systems.

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