Multi-Stability in Conditional Symmetric Systems

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1 Introduction

Some of the coexisting attractors in symmetric or asymmetric systems share the geometric structure even with unified Lyapunov exponents [1–10]. In symmetric systems, many of the coexisting attractors stand in separate subspace respecting to some of the coordinate-axis [1–7]. Coexisting attractors may be embedded in each other in asymmetric systems [8–10]. Dynamical systems may exhibit coexisting attractors with other relative positions, specifically some of which in phase space may have some property of symmetry according to a super-plane but need extra offset boosting. The symmetry mentioned above is obtained from a necessary transformation and therefore, is defined as conditional symmetry. As shown in Fig. 1, an asymmetric system can be revised to be symmetric one by polarity revise in the feedback term while conditional symmetric system needs a function-based polarity balance.

Any transformation in a dynamical system should obey the basic law of polarity balance to maintain its basic dynamical properties. A symmetric system keeps its polarity balance even when some of the variables get polarity reversed. More generally, the polarity imbalance in a dynamical system can be induced by the polarity reversal of any of the variables, or from the function in the feedback, or even from the time dimension. Polarity reverse from some of the variables can retain its polarity balance in a symmetric system while an asymmetric one loses its balance of polarity. It shows that the polarity balance can be restored in some specific asymmetric systems when some of the variables are offset boosted in the feedback function where the offset boosting does not change the polarity of the left-hand side of the differ-

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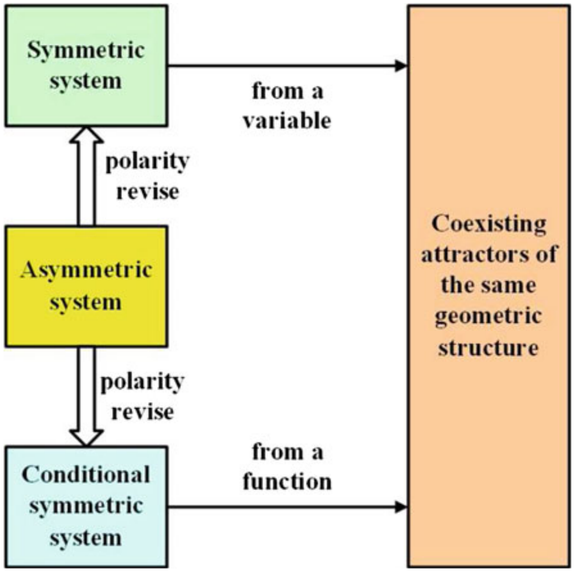

Fig. 1 System structure connected with symmetry

ential equation but give a negative sign on the right-hand side. In the following, we discuss the conception of conditional symmetry and the approach for constructing a conditional symmetric system with newly found examples. Here, the fundamental factor is polarity control. This control sometimes is combined with other control like frequency control, amplitude control making the attractor(s) with different geometric and distribution characteristics.

2 Conception of Conditional Symmetry

Offset boosting is the key factor for understanding conditional symmetry since it does not bring any change with derivative operation but can bring a negative sign in a nonmonotonic function. For a dynamical system $\dot{X} = F(X)$ $(X = (x_1, x_2, ..., x_N))$, a variable substitution of $x_j \to x_j + d_j$ (here, $1 \leqslant j \leqslant N, i \in \{1, 2, ..., N\} \setminus \{j\}$ makes the variable x_j in system $\dot{X} = F(X)$ exhibit **offset boosting**, which means that the average of the variable x_j is boosted by the new introduced constant d_j. Specifically, if the above substitution only introduces a separate constant d_j in one dimension on the right-hand side of the equations, then the system can be regarded as a **offset-boostable system** [11]. In electrical circuit, the variable x_j represents a circuit signal, and the newly introduced constant d_j is a direct current source. Similarly, a dynamical system $\dot{X} = F(X) = (f_1(X), f_2(X), ..., f_N(X))$ $(X = (x_1, x_2, ..., x_N))$ can be defined as an **N-D offset-boostable system** [11–14] if there exist n variable substitutions recover its governing equation, except for n additional constants allowing offsets with those variables. As reported in Ref. [14], the nonmonotonic function like an absolute value function or a trigonometric function may result in a polarity reversal, which can be applied to reconstruct polarity balance for conditional symmetry.

For a differential equation $\dot{X} = F(X) = (f_1(X), f_2(X), ..., f_N(X))$ $(X = (x_1, x_2, ... , x_N))$, if there exists a variable substitution including polarity reversal and offset boosting like $u_{i_1} = -x_{i_1},\ u_{i_2} = -x_{i_2}, ..., u_{i_k} = -x_{i_k},\ u_{j_1} = -x_{j_1} + d_{j_1},\ u_{j_2} = -x_{j_2} + d_{j_2}, ...,\ u_{j_l} = -x_{j_l} + d_{j_l}, u_i = x_i$ (here, $1 \leqslant i_1, ..., i_k \leqslant N,\ 1 \leqslant j_1, ..., j_l \leqslant N,\ i_1, ..., i_k$and $j_1, ..., j_l$ are not identical, $i \in \{1, 2, ..., N\} \setminus \{i_1, ..., i_k, j_1, ..., j_l\}$), the derived equation retains its balance of polarity on the two sides of the equation and satisfies $\dot{U} = F(U)$ $(U = (u_1, u_2, ..., u_N))$. The corresponding system $\dot{X} = F(X)$ $(X = (x_1, x_2, ..., x_N))$ is defined as one of l-dimensionally conditional symmetry, since the polarity balance needs l-dimensional offset boosting [12, 13]. Specifically, for a three-dimensional dynamical system, $\dot{X} = F(X)$ $(X = (x_1, x_2, x_3))$, there exist only conditional rotational symmetry in 1-dimension and conditional reflection symmetry in 1-dimension or 2-dimension. Note that offset boosting does not produce a minus sign on the left-hand side of the equation, since $d(x_{jm} + d_{jm}) = d(x_{jm})$.

3 Constructing Conditional Symmetry from Single Offset Boosting

A jerk system is a simple structure to pass the polarity, which consequently provides an easy way to consider polarity balance. Such a case can be found even in a hypogenetic chaotic jerk flow JH5 with absolute value nonlinearities [15], as follows:

$$\begin{cases} \dot{x} = |y| - b, \\ \dot{y} = z, \\ \dot{z} = f(x) - y - az. \end{cases} \tag{1}$$

When $a = 0.6,\ b = 1,\ f(x) = |x| - c,\ c = 2$, the system (1) is chaotic and has two coexisting symmetric attractors, with Lyapunov exponents (0.0534, 0, −0.6534) and Kaplan-Yorke dimension $D_{KY} = 2.0817$, as shown in Fig. 2. System (1) has four equilibrium points, $(\pm 3, 1, 0)$, $(\pm 1, -1, 0)$, which may consist of the centers of different attractors. The symmetric coexistence of chaotic solutions can be explained by a transformation. Even the structure of system (1) has no symmetry, a variable boosting in the variable x can return a symmetry-like transformation, which is now defined as conditional symmetry and correspondingly give birth to symmetric bistability. Specifically, let $x = u + d,\ y = -v,\ z = -w$(here, d is a new introduced constant). If the condition $c = (|x| + |u + d|)/2$ is satisfied, the new deduced system has a conditional rotational symmetry because the symmetry-like transformation $\dot{(u)} = |v| - b, \dot{(v)} = w, \dot{(w)} = -|u + d| - v - aw + c$ can be obtained. In the new

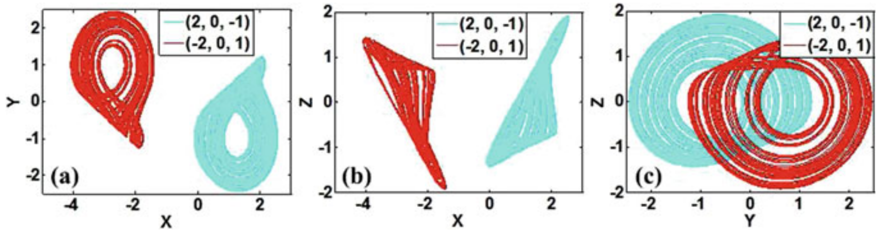

Fig. 2 Symmetric strange attractors for system (1) with $a = 0.6,\ b = 1,\ c = 2$. **a** x-y plane. **b** x-z plane. **c** y-z plane

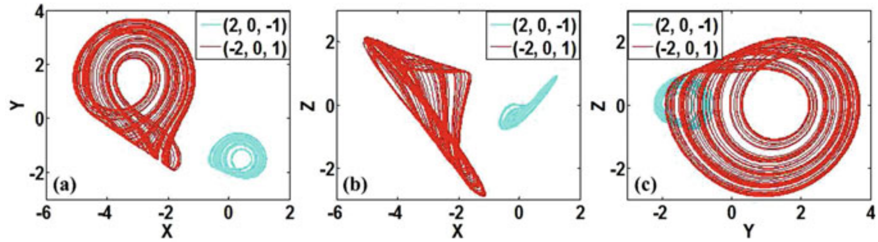

Fig. 3 Asymmetric strange attractors for system (1) with $a = 0.6,\ b = 1.5,\ c = 2$. **b** x-y plane. **b** x-z plane. **c** y-z plane

space of variables u, v, w, the newly derived equations are conditionally identical to the original one, when $c - |u + d| = |x| - c$. Here, the symmetry is obtained on the condition equation in the z-dimension. The basins of attraction are shown in Fig. 4a, as predicted, although the regions in light blue and red representing two different attractor basins are asymmetric, the strange attractors, represented in cross section by black lines, are symmetric and nearly touch their basin boundaries. Two of the equilibrium points are within the attracting basins, while the other two are on the basin boundary. As shown in Fig. 3, when $b = 1.5$, two asymmetric strange attractors coexist, whose attracting basins are shown in Fig. 4b. Here, the black lines of cross section are asymmetric.

A similar chaotic system of conditional rotational symmetry can be obtained if the absolute value function of y is replaced by a quadratic function rewritten as

$$\begin{cases} \dot{x} = y^2 - a, \\ \dot{y} = bz, \\ \dot{z} = -y - z + f(x). \end{cases} \tag{2}$$

The revised system keeps on the polarity balance required for the symmetry transformation from the polarity reversal of the function given by offset boosting. The system (2) yields coexisting symmetric attractors according to the y-axis and z-axis, as shown in Fig. 5. System (2) has two groups of equilibria with the same stability when $a = 1.22,\ b = 8.48,\ f(x) = |x| - 3$, which are $P_{11} = (4.1045, 1.1045, 0),\ P_{12} =$

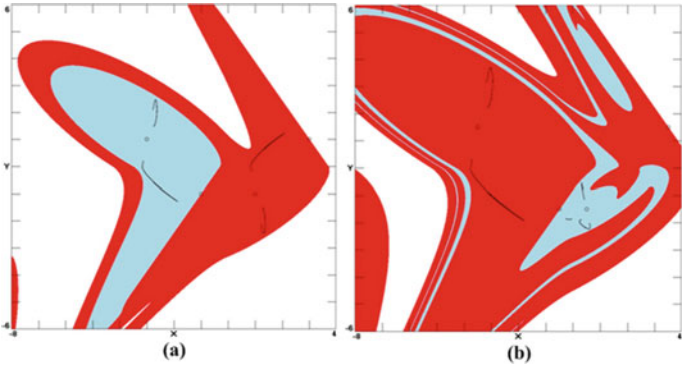

Fig. 4 Cross section $z = 0$ of the basins of attraction for the symmetric strange attractors of system (5). **a** $a = 0.6,\ b = 1,\ c = 2$. **b** $a = 0.6,\ b = 1.5,\ c = 2$.

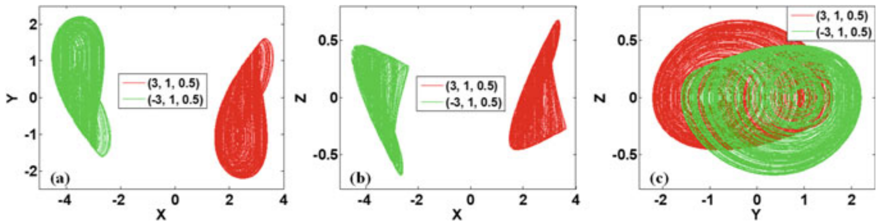

Fig. 5 Symmetric strange attractors for system (2) with $a = 1.22,\ b = 8.48,\ f(x) = |x| - 3$. **a** x-y plane. **b** x-z plane. **c** y-z plane

(1.8955, 1.1045, 0), $P_{21} = (1.8955, 1.1045, 0)$, $P_{22} = (4.1045, 1.1045, 0)$. When $|u + c| - 3 = 3 - |x|$, the transformation $x \to u + c$, $y \to -v$, $z \to -w$ in Eq. (2) is subject to the same governing equation and therefore generates coexisting rotational symmetric attractors. The basin of attraction for the coexisting attractors shows that the symmetric attractors lie in corresponding asymmetric basins [14].

As another case, we can construct a conditional reflection symmetric system by introducing the absolute value function for returning a polarity balance based on offset boosting [14], as follows:

$$\begin{cases} \dot{x} = y^2 - az^2, \\ \dot{y} = -z^2 - by + c, \\ \dot{z} = yz + f(x). \end{cases} \tag{3}$$

System (3) also has two groups of equilibria, which are P_{11}=(6.5576, 1.5, 2.3717), $P_{12} = (1.9881, 0.8, 1.2649)$, $P_{13} = (4.0119, 0.8, 1.2649)$, with eigenvalues $\lambda_{11} = (0.6029, 1.9265 \pm 3.7926i)$, $\lambda_{12} = (1.5526, 0.3013 \pm 1.9123i)$, and λ_{13}=(1.4116, $1.1808 \pm 1.6515i$), respectively, $P_{21} = (6.5576, 1.5, 2.3717)$, $P_{22} = (4.0119, 0.8, 1.2649)$, $P_{23} = (1.9881, 0.8, 1.2649)$, with the same eigenvalues $\lambda_{21} = (1.1397, 1.0551 \pm 2.9085i)$, $\lambda_{22} = (1.5526, 0.3013 \pm 1.9123i)$, and λ_{23}=(1.4116, 1.1808±

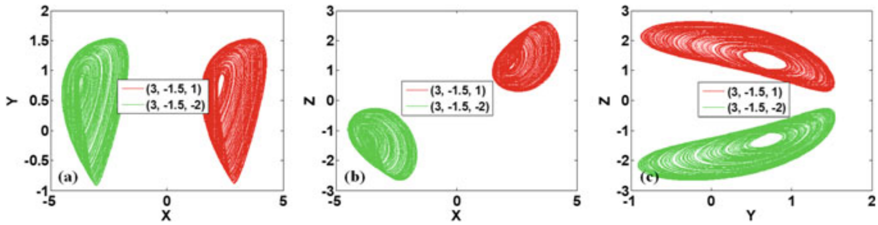

Fig. 6 Symmetric strange attractors for system (3) with $a = 0.4,\ b = 1.75,\ c = 3,\ f(x) = |x| - 3$. **a** x-y plane. **b** x-z plane. **c** y-z plane

$1.6515i$), respectively. The equilibrium point P_{21} is also a stable focus, and therefore it coexists with the other strange attractors [14]. The conditional symmetry of Eq. (3) is obtained by the offset boosting of the variable x resulting in a polarity reversal $|u + c| - 3 = 3 - |x|$, which completes polarity balance for the conditional symmetry transformation $x \to u + c,\ y \to v,\ z \to -w$. Moreover, the conditional symmetry can also be constructed by introducing other non-monotonic functions $F(x)$, such as $F_2(x) = 3 - |x|$, $F_3(x)$= 1.5sin(x), and $F_4(x)$= 1.5cos(x). Since the trigonometric functions sin(x) and cos(x) are also periodic, the attractor basin will be correspondingly periodic giving two groups of infinitely many duplication of the attractors [13].

4 Constructing Conditional Symmetry from Multiple Offset Boosting

More generally, conditional symmetry can be constructed from a structure with multiple offset boosting in addition to rigid variable-boostable systems [11]. Let us think, if the goal is to construct a conditional reflection symmetric system given the polarity reversal in the variable x_i, the polarity revise of the dimension of x_i on the left-hand side of x_i in turn requires the adjustment in the polarity with the terms on the right-hand side of x_i to get $-f_i(X)$. At the same time, all the other dimensions should keep the polarity balance regardless of that $-x_i$ may introduce a minus sign on the right-hand side, which further requires a new polarity reverse from offset boosting to cancel it. Here, the polarity balance of a polynomial equation $f_j(X)$ is saved by introducing new functions in $f_j(x_1, x_2, ..., -x_i, ..., F_{j_1}(x_{j_1}), F_{j_2}(x_{j_2}), ..., F_{j_l}(x_{j_l}), ..., x_N)$ $(1 \leqslant j_1, ..., j_l \leqslant N,\ j_1, ..., j_l$ are not identical, and l is an odd) for variables $(x_{j_1}, x_{j_2}, ..., x_{j_l})$, which admits offset boosting returning a new minus sign to cancel the one from $-x_i$, by $F_{jm}(x_{jm} + d_{jm} = -F_{jm})(x_j m)$ $(j_1 \leqslant j_m \leqslant j_l)$.

Four general equations for hosting conditional symmetry are designed for exhaustive computer searching [12]. We can conclude that Eqs. (4) and (5) can be transformed to exhibit conditional symmetry from 1D offset boosting, while Eqs. (6) and (7) can be modified to show conditional symmetry from 2D offset boosting. Equa-

Table 1 Conditional symmetric systems (CSS)

Systems	Equations	Parameters	LEs	D_{KY}	(x0,y0,z0)
CSS1	$\dot{x} = F(y)$, $\dot{y} = z$, $\dot{z} = -x^2 - az + b(F(y))^2 + 1$, $F(y) = \|y\| - 4$	a= 2.6, b= 2	0.0463, 0 −2.6463	2.0175	0.5, 4, −1
CSS2	$\dot{x} = y$, $\dot{y} = F(z)$, $\dot{z} = x^2 - ay^2 + bxy + xF(z)$ $F(z) = \|z\| - 8$	a= 1.24, b= 1	0.0463, 0, −2.6463	2.0513	4, 0.8, −2
CSS3 (from VB6)	$\dot{x} = 1 - G(y)z$, $\dot{y} = az^2 - G(y)z$, $\dot{z} = F(x)$, $F(x) = \|x\| - 3$ $G(y) = \|y\| - 5$	a= 0.22	0.0729 0 −1.6732	2.0436	−1, 1, −1
CSS4	$\dot{x} = F(y)$, $\dot{y} = xG(z)$, $\dot{z} = -axF(y) - bxG(z) - x^2 + (F(y)^2)$ $F(y) = \|y\| - 5$ $G(z) = \|z\| - 5$	a= 3, b= 1.2	0.0506 0 −0.2904	2.1735	0, −6, −6

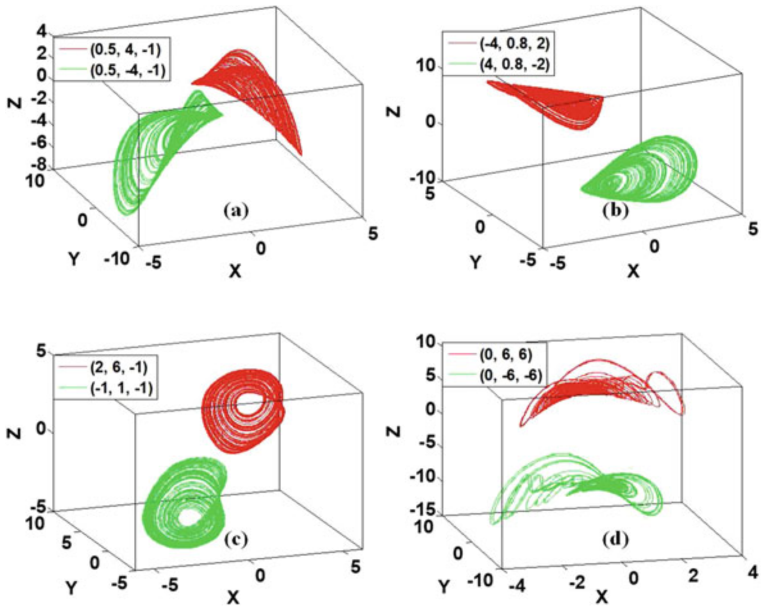

Fig. 7 Coexisting attractors. **a** CSS1 induced by 1D offset boosting in the y-dimension. **b** CSS2 induced by 1D offset boosting in the z-dimension. **c** CSS3 induced by 2D offset boosting in the x and y dimensions. **d** CSS4 induced by 2D offset boosting in the y and z dimensions

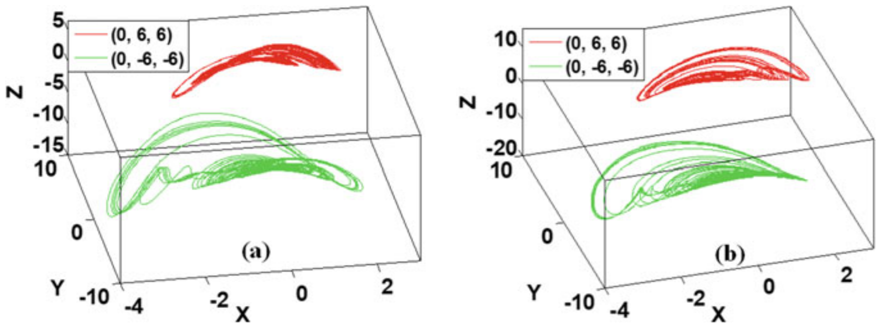

Fig. 8 Coexisting attractors in CSS4 induced by 2D offset boosting in the y and z dimensions. **a** $F(y) = |y| - 5$, and $G(z) = |z| - 3.3$. **b** $F(y) = |y| - 4.5$, and $G(z) = |z| - 5$

tions (4) and (5) are different from the above mentioned cases reported in Ref. [14], where a single 1D offset boosting is in the third dimension. Here, the new conditional symmetry comes from a jerk structure, where the reflection or rotational symmetry is broken by a neighbor variable, but the polarity balance can be restored by a general offset boosting. Equations (6) and (7) are the structures for hosting conditional reflection symmetry, where 2D offset boosting is necessary for restoring polarity bal-

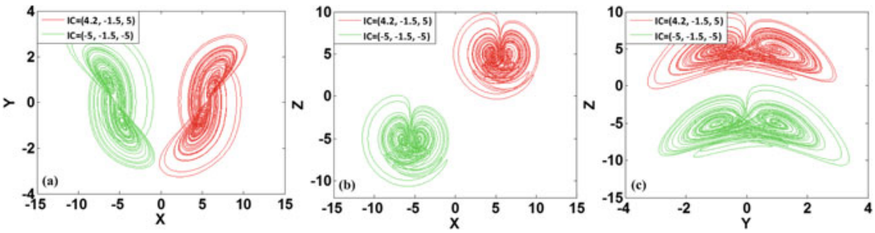

Fig. 9 Coexisting attractors for system (9) with $F(x) = |x| - 5.5,\ G(z) = |z| - 5$. **a** x-y plane. **b** x-z plane. **c** y-z plane. Red is for $IC = (4.2, -1.5, 5)$ and green is for $IC = (-5, -1.5, -5)$

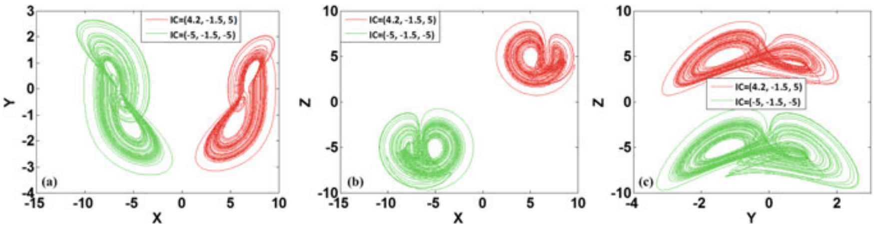

Fig. 10 Coexisting attractors for system (9) with $F_1 = |x| - 6.6,\ F_2 = |x| - 6$. **a** x-y plane. **b** x-z plane. **c** y-z plane. Red is for $IC = (4.2, -1.5, 5)$ and green is for $IC = (-5, -1.5, -5)$

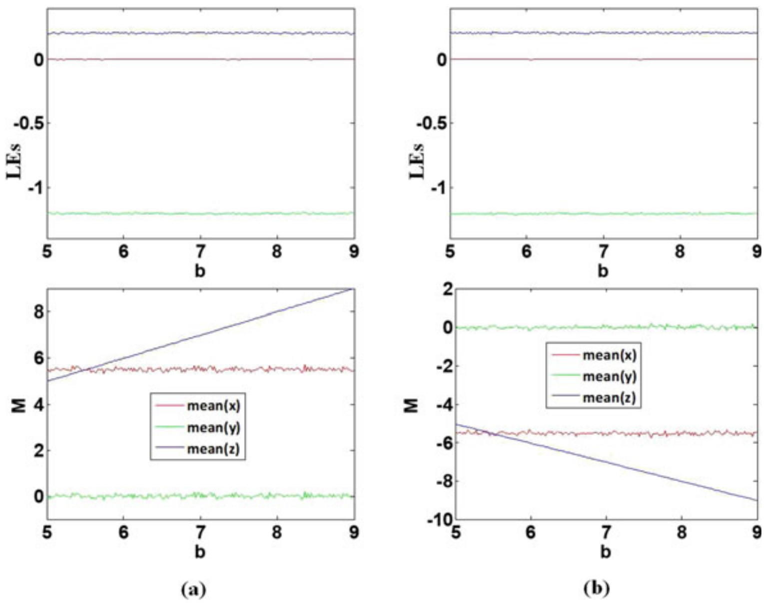

Fig. 11 Lyapunov exponents and average value for system (9) with $F(x) = |x| - 5.5,\ G(z) = |z| - b$. **a** $IC = (4.2, -1.5, 5 + b)$. **b** $IC = (-5, -1.5, -5 - b)$

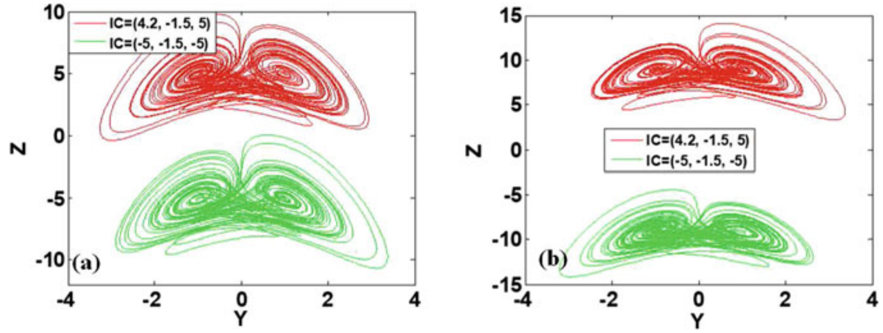

Fig. 12 Coexisting attractors for system (9) in the y–z plane. Red is for $IC = (4.2, -1.5, 5)$ and green is for $IC = (-5, -1.5, -5)$. **a** $F(x) = |x| - 5.5,\ G(z) = |z| - 5$. **b** $F(x) = |x| - 5.5,\ G(z) = |z| - 9$

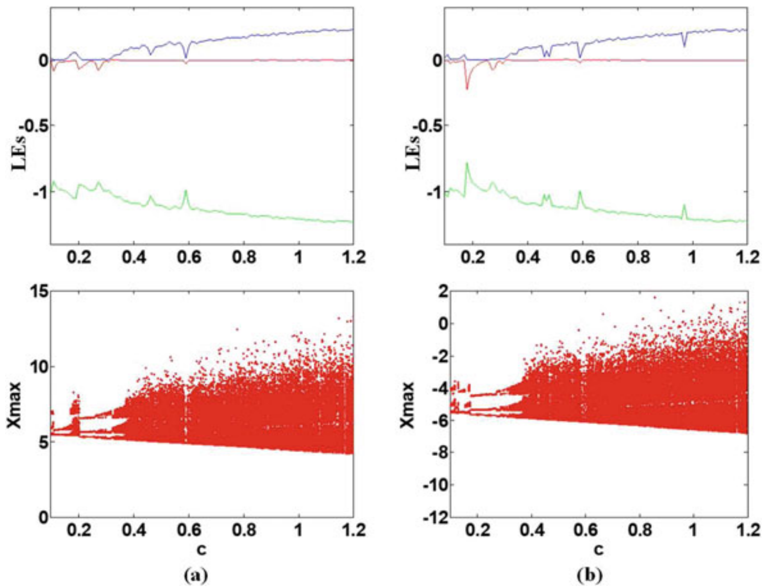

Fig. 13 Bifurcation diagrams and Lyapunov exponents of system (9) with $F(x) = |x| - 5.5,\ G(z) = |z| - 5$, when c is varied in [0.1, 1.2]. **a** $IC = (4.2, -1.5, 5)$. **b** $IC = (-5, -1.5, -5)$

ance. Newly constructed conditional symmetric systems (CSS) are listed in Table 1, exhibiting chaotic coexisting attractors.

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = a_1xy + a_2x^2 + a_3y^2 + a_4z + a_5z^2 + a_6. \end{cases} \tag{4}$$

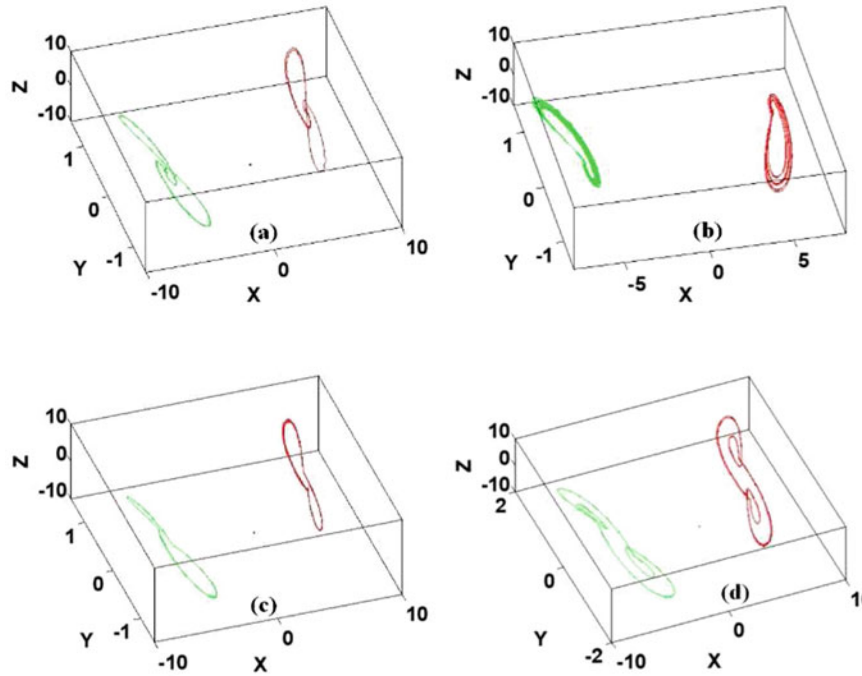

Fig. 14 Some other coexisting attractors of system (9) with $F(x) = |x| - 5.5,\ G(z) = |z| - 5$, where $IC = (4.2, -1.5, 5)$ is red, $IC = (-5, -1.5, -5)$ is green. **a** $c = 0.1$. **b** $c = 0.17$. **c** $c = 0.3$. **d** $c = 0.59$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = a_1x^2 + a_2y^2 + a_3xy + a_4xz + a_5yz + a_6z^2 + a_7. \end{cases} \tag{5}$$

$$\begin{cases} \dot{x} = a_1z^2 + a_2xz + a_3yz + a_4, \\ \dot{y} = a_5z^2 + a_6xz + a_7yz + a_8, \\ \dot{z} = a_9z + a_{10}x + a_{11}y. \end{cases} \tag{6}$$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = xz, \\ \dot{z} = a_1xy + a_2xz + a_3x^2 + a_4y^2 + a_5z^2 + a_6yz + a_7. \end{cases} \tag{7}$$

Symmetric pairs of coexisting attractors in the conditional symmetric versions are shown in Fig. 7. Figure 7a shows the coexisting attractors in CSS1 induced by 1D offset boosting in the y-dimension, while Fig. 7b plots the coexisting symmetric attractors in the plane $x = 0$, requiring an offset boosting in the z-dimension. Two symmetric attractors in the plane $z = 0$ in system CSS3 demand offset boosting in the x and y dimensions, as indicated in Fig. 7c. In Fig. 7d, two symmetric attractors in the plane $x = 0$ in system CSS4 require offset boosting in the y and z dimensions. A suitable threshold in the non-monotonic operation $F(.)$ is necessary for restoring the polarity balance under offset boosting. For $F(x) = |x| - a$, to obtain $F(x + 2a) = |x + 2a| - a = -F(x)$ (or, $F(x - 2a) = |x - 2a| - a = -F(x)$), the variable x should be in the region $[-2a\ , 0]$ (or, $[0, 2a]$) ($a \geqslant 0$). Modifying the threshold can give coexisting asymmetric attractors [12], as shown in Fig. 8.

5 Constructing Conditional Symmetric System from Revised Polarity Balance

It is found that even a symmetric system [16–20] is possible to revise the polarity balance to obtain the conditional symmetry. An easy example is from the rotational symmetry in the Sprott B system:

$$\begin{cases} \dot{x} = yz, \\ \dot{y} = x - y, \\ \dot{z} = 1 - xy. \end{cases} \tag{8}$$

Here, the polarity balance maintains from $-x$ and $-y$ for the symmetric structure. Meanwhile, the unique structure can still maintain its polarity balance when the polarity is changed from an extra function without revising its basic dynamics, as

$$\begin{cases} \dot{x} = G(z)y, \\ \dot{y} = F_1(x) - y, \\ \dot{z} = c - F_2(x)y. \end{cases} \tag{9}$$

When $F_1(x) = F_2(x) = F(x) = |x| - 5.5,\ G(z) = |z| - 5,\ c = 1$, system (9) exhibits coexisting conditional reflection symmetric attractors with Lyapunov exponents (0.2101, 0, −1.2101) and Kaplan-Yorke dimension $D_{KY} = 2.173$, as shown in Fig. 9.

Mismatched offset boosting may result in asymmetric coexisting attractors. In system (8), the variable x appears twice, and so two functions are necessary for conditional symmetry. Moreover, these two functions should keep phase synchronization for returning the same polarity reverse. As shown in Fig. 10, when $F_1 = |x| - 6.6$, $F_2 = |x| - 6$ and $G(z) = |z| - 5$, two asymmetric strange attractors appear, as predicted. Offset controllers in different variables may keep independent from each other, as shown in Fig. 11. When $F(x) = |x| - 5.5$, $G(z) = |z| - b$ and b is varied, as in Ref. [5, 9], the average of the variable z changes smoothly without revising the averages of other variables x and y with the same Lyapunov exponents. As shown in Fig. 12, larger offset controller b makes the symmetric pair of coexisting attractors farther in the z-dimension.

Normal bifurcation parameter c in system (9) control the dynamical behavior dramatically. But in the special structure of conditional symmetry, system (9) displays coexisting bifurcations from different initial conditions, as shown in Fig. 13. The common bifurcation controller c may lead to different evolutions, giving asymmetric coexisting attractors, as shown in Fig. 14.

6 Discussions and Conclusions

Multi-stability shows that various profiles, specifically symmetric strange attractors and asymmetric ones, can coexist in asymmetric basins of attraction within a simple nonlinear structure due to conditional symmetry. Offset boosting is the key routine for restoring polarity balance and thereafter produces symmetric pairs of coexisting attractors. Offset-boostable system provides a compact structure for coining conditional symmetric system, while multiple dimensional offset boosting can find more cases of chaotic systems of conditional symmetry. Mismatched offset boosters may draw the system to exhibit asymmetric coexisting attractors. The symmetric system can also be transformed into a conditional symmetric one if the polarity balance is restored based on offset boosting.

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References

1. C. Li, J.C. Sprott, W. Thio, Linearization of the Lorenz system. Phys. Lett. A **379**(10–11), 888–893 (2015)
2. L. Zhou, C. Wang, L. Zhou, Cluster synchronization on multiple sub-networks of complex networks with nonidentical nodes via pinning control. Nonlinear Dyn. **83**(1–2), 1079–1100 (2016)
3. C. Li, I. Pehlivan, J.C. Sprott, A. Akgul, A novel four-wing strange attractor born in bistablity. IEICE Electron. Express **12**(4), 20141116 (2015)
4. J.C. Sprott, Simplest chaotic flows with involutional symmetries. Int. J. Bifurc. Chaos **24**(1), 1450009 (2013)
5. J.C. Sprott, New chaotic regimes in the Lorenz and Chen systems. Int. J. Bifurc. Chaos **25**(5), 1550033 (2015)
6. Q. Lai, S. Chen, Generating multiple chaotic attractors from Sprott B system. Int. J. Bifurc. Chaos **26**(11), 1650177 (2016)
7. Q. Lai, T. Nestor, J. Kengne, X.W. Zhao, Coexisting attractors and circuit implementation of a new 4d chaotic system with two equilibria. Chaos, Solitons Fractals **107**, 92–102 (2018)
8. R. Barrio, F. Blesa, S. Serrano, Qualitative analysis of the Rössler equations: Bifurcations of limit cycles and chaotic attractors. Physica D **238**(13), 1087–1100 (2009)
9. E. Freire, E. Gamero, A.J. Rodríguez-Luis, A. Algaba, A note on the triple-zero linear degeneracy: normal forms, dynamical and bifurcation behaviors of an unfolding. Int. J. Bifurc. Chaos **12**(12), 2799–2820 (2002)

10. J.C. Sprott, C. Li, Asymmetric bistability in the Rössler system. Acta Phys. Pol., B **48**(1), 699–792 (2017)
11. C. Li, J.C. Sprott, Variable-boostable chaotic flows. Optik: Int. J. Light Electron Optics **127**(22), 10389–10398 (2016)
12. C. Li, J.C. Sprott, Y. Liu, Z. Gu, J. Zhang, Offset boosting for breeding conditional symmetry. Int. J. Bifurc. Chaos **28**(14), 1850163 (2018)
13. C. Li, J.C. Sprott, W. Hu, Y. Xu, Infinite multistability in a self-reproducing chaotic system. Int. J. Bifurc. Chaos **27**(10), 1750160 (2017)
14. C. Li, J.C. Sprott, H. Xing, Constructing chaotic systems with conditional symmetry. Nonlinear Dyn. **87**(2), 1351–1358 (2017)
15. C. Li, J.C. Sprott, H. Xing, Hypogenetic chaotic jerk flows. Phys. Lett. A **380**(11–12), 1172–1177 (2016)
16. C. Li, W. Hu, J.C. Sprott, X. Wang, Multistability in symmetric chaotic systems. Eur. Phys. J.: Special Topics **224**(8), 1493–1506 (2015)
17. K. Rajagopal, S. Panahi, A. Karthikeyan, A. Alsaedi, V.-T. Pham, T. Hayat, Some new dissipative chaotic systems with cyclic symmetry. Int. J. Bifurc. Chaos **28**(13), 1850164 (2018)
18. X. Di, J. Li, H. Qi, L. Cong, H. Yang, A semi-symmetric image encryption scheme based on the function projective synchronization of two hyperchaotic systems. PLoS ONE **12**(9), e0184586 (2017)
19. X. Wang, A. Akgul, S. Cicek, V.-T. Pham, D.V. Hoang, A chaotic system with two stable equilibrium points: Dynamics, circuit realization and communication application. Int. J. Bifurc. Chaos **27**(08), 1750130 (2017)
20. J. Liu, J.C. Sprott, S. Wang, Y. Ma, Simplest chaotic system with a hyperbolic sine and its applications in DCSK scheme. IET Commun. **12**(7), 809–815 (2018)