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Investigating chaotic attractor of the simplest chaotic system with a line of equilibria

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Abstract. A new three-dimensional chaotic flow is proposed in this paper. The system is the simplest chaotic flow that has a line of equilibria. The chaotic attractor of the system is very special with two slow and fast parts. In other words, the dynamic of the system is a combination of slow and fast states. The unique chaotic attractor of the system is investigated. Dynamical properties of the system, such as stability of equilibrium points and bifurcation diagrams, are studied. We believe that such a system with these special properties is proposed for the first time in this paper.

1 Introduction

Chaotic flows have a mysterious in the generation of chaotic attractors. For many years there was a hypothesis that chaotic attractors are related to saddle equilibria [1,2]. In 2011, a chaotic system without any equilibrium point was proposed [3]. It was the first example that infracted the old hypothesis. Many other counterexamples were proposed after that [4,5]. Chaotic systems with a line of equilibria [6,7], chaotic systems with a surface of equilibria [8], and chaotic flows with a plane of equilibria [9] were some examples. To understand the chaotic dynamics, various groups of chaotic attractors have been studied [10,11]. A chaotic system with total amplitude control has been constructed in [12]. This method has involved an infinite line of equilibria. Controlling chaotic dynamics has been a hot topic [13].

Attractors can be categorized in hidden or self-excited attractors [14]. Hidden attractors are not associated with a saddle point equilibria, while self-excited attractors are [15]. It means that hidden attractors only can be found using numerical searches [16]. Hidden chaotic attractors has been a hot topic [17,18]. Multistability is an important feature of dynamical systems [19,20]. Multistability can cause a huge shift from one attractor to another just by a small perturbation. Many studies have been done on chaotic systems with multistability [21,22]. Multistability of a hyperchaotic system with a line of equilibria has been discussed in [23].

In [24], three conditions were proposed to have a standard in the publication of chaotic dynamics. The first one was that the system should credibly model some important unsolved problems in nature. The second one was that the dynamic of

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the system should be previously unobserved. The third condition is that the system should be simpler than other known examples, which can show the dynamic. In this paper, the simplest system with a line of equilibria is proposed. The system shows some special features which are discussed in the following of the paper. In Section 2, the system is proposed. The chaotic dynamic of the system is discussed in Section 3. Also, the bifurcation analysis of the system is studied in this section. The paper is concluded in Section 4.

2 The proposed system

The new three-dimensional chaotic flow is as follows:

$$\begin{aligned} \dot{x} &= ay \\ \dot{y} &= xz \\ \dot{z} &= y - z - y^2. \end{aligned} \tag{1}$$

The system shows an elegant chaotic attractor in parameter $a = 289$ and initial conditions $(x_0, y_0, z_0) = (32, 0.1, 0)$. Three projections of the chaotic attractor and three-dimensional chaotic attractor of system (1) are shown in Figure 1. Lyapunov exponents of the chaotic attractor are $(0.0633, 0, -1.0633)$. The probability P at a distance r from the D-dimensional strange attractor lies within the basin of attraction can be calculated to find the basin size of the strange attractor [25]. The probability P of the chaotic attractor of system (1) is $P \sim 130/r^{1.7}$ with the limit of large r. So the basin of this attractor is Class 3 which means it extends to infinity in some directions while it occupies an ever-decreasing fraction of the state space. The blue line in Figure 1 shows the projection of the line of equilibria.

3 Dynamical properties of the proposed system

In this section, the dynamical properties of the simplest chaotic flow with a line of equilibria are investigated.

3.1 Equilibrium points and their stabilities

The first step of investigating the dynamical properties of system (1) is calculating equilibrium points and their stabilities. Setting zero the right-hand side of equation (1) depicts that the system has a line of equilibria at $\mathrm{Eq} = (x, 0, 0)$. Projections of the line of equilibria are shown in blue color in Figure 1. To calculate the stability of the line of equilibria, the Jacobian matrix of system (1) at the equilibrium point is calculated as equation (2).

$$\mathrm{Jac}|_{\mathrm{Eq}} = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & x \\ 0 & 1 & -1 \end{bmatrix}. \tag{2}$$

The characteristic equation is $\lambda^3 + \lambda^2 - x\lambda = 0$, so parameter a does not have any effect on the stability of the line of equilibria. Real and Imaginary parts of Eigenvalues of system (1) for each point of the line of equilibria are shown in Figure 2 in the interval $x \in [-200, 200]$. The real part of Eigenvalues shows that the line of

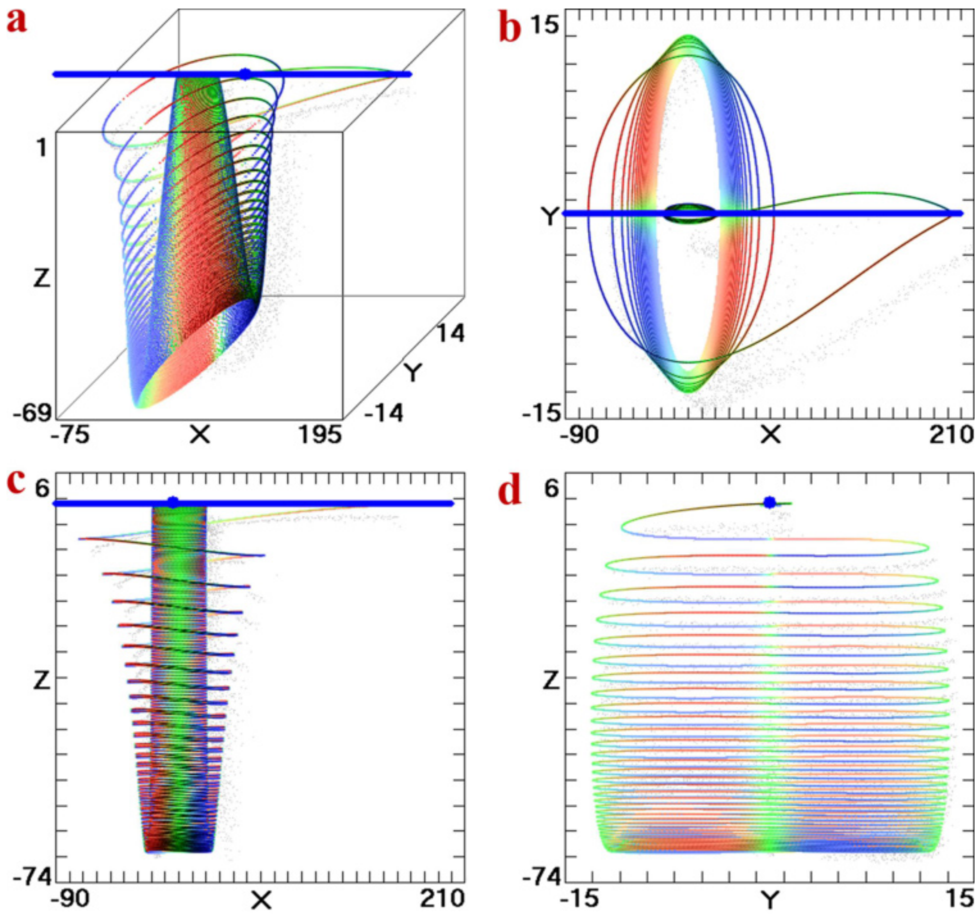

Fig. 1. Projections of chaotic attractor of system (1) in parameter $a = 289$ and initial conditions $(x_0, y_0, z_0) = (32, 0.1, 0)$. (a) In $X{-}Y{-}Z$ space; (b) in $X{-}Y$ plane; (c) in $X{-}Z$ plane; (d) in $Y{-}Z$ plane.

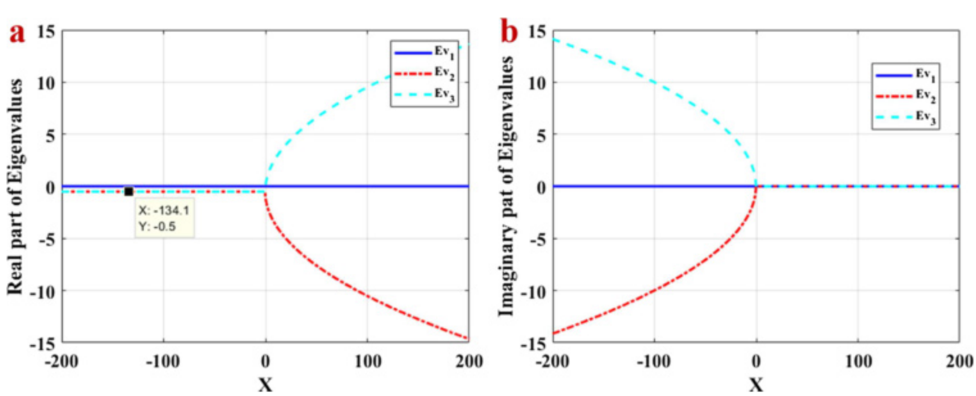

Fig. 2. Eigenvalues of system (1) at the origin. (a) Real part of eigenvalues. (b) Imaginary part of eigenvalues.

equilibria is unstable in positive x. However, the stability of the line of equilibria in negative x cannot be determined using the Eigenvalues. Numerical studies show that the line of equilibria is stable in negative values of x since there are some initial conditions which make the dynamic to be attracted to the line of equilibria. So the system is multistable with two equilibrium point and chaotic dynamics. The stable part of the line of equilibria is spiral. Every dynamics of system (1) is dissipative since trac(Jac) $= -1$, which is not dependent on states or parameter.

3.2 The unique dynamic of the chaotic attractor

The chaotic dynamic of the system has a particular property. Three time series of the states x, y, and z variables for the chaotic attractor are shown in part a of Figure 3 with black, cyan, and magenta color, respectively. The chaotic dynamic of system (1) is hidden since it has a line of equilibria. The system has a unique property, which can be seen in the time series of the chaotic attractor of the system. Variable z plays the rule of a switch variable in this dynamic. Parts b and c of Figure 3 shows a zoomed view of time series in two parts of a cycle of variable z: positive part and negative part. For positive z, x and y grow exponentially. If we consider a positive constant value of variable z (e.g., $z = 0.06$), the dynamic of two coupled x and y are shown in Figure 4. After that, they force z to become negative because of the y^2 term, and the two variables x and y then form a harmonic oscillator with a frequency given by $\sqrt{-az}$. If we consider a negative constant value of variable z (e.g., $z = -0.06$), the dynamic of two coupled variables x and y are shown in Figure 5. The variable z is a kind of relaxation oscillator that parametrically drives the frequency of the x, y sinusoidal oscillator. Figure 3 shows that increasing the variable z causes a downward sweep in frequency. In other words, the slow variable z plays the rule of control parameter, which drives the oscillation of fast variables x and y.

To better understand the dynamics of the system in two positive and negative parts of the variable z, consider the simplified system as equation (3).

$$\begin{array}{l} \dot{x} = ay \\ \dot{y} = z^* x \end{array}. \tag{3}$$

The system has an equilibrium point in origin. Jacobian matrix of the system is,

$$\text{Jac} = \begin{bmatrix} 0 & a \\ z^* & 0 \end{bmatrix}. \tag{4}$$

The characteristic equation of the system is $\lambda^2 - az^* = 0$. So the system has two Eigenvalues at $\lambda = \pm\sqrt{az^*}$. In this study, $a = 289$ and z^* can be positive or negative. In positive z^*, the system has one positive and one negative Eigenvalue, so the origin is unstable. In negative z^*, there are two pure imaginary Eigenvalues as $\lambda = \pm i\sqrt{-az^*}$. So origin is a center point, which means the dynamic of the system is a cycle, and the time series are harmonic oscillator with frequency $\sqrt{-az^*}$. In other words, system (3) can be solved as follows:

$$\frac{dx}{dy} = \frac{ay}{z^* x} \to z^* \left(x^2 - x_0^2\right) = a\left(y^2 - y_0^2\right) \to z^* x^2 - ay^2 = z^* x_0^2 - ay_0^2. \tag{5}$$

If z^* is negative, then the solution is ellipsoid, and if z^* is positive, the solution is hyperbolic. So the solution is a bounded harmonic oscillator when z^* is negative, and the solution is unbounded when z^* is positive. Figure 4 shows the time series and state-space of system (3) in $z^* = 0.06$. The dynamic of the system exponentially

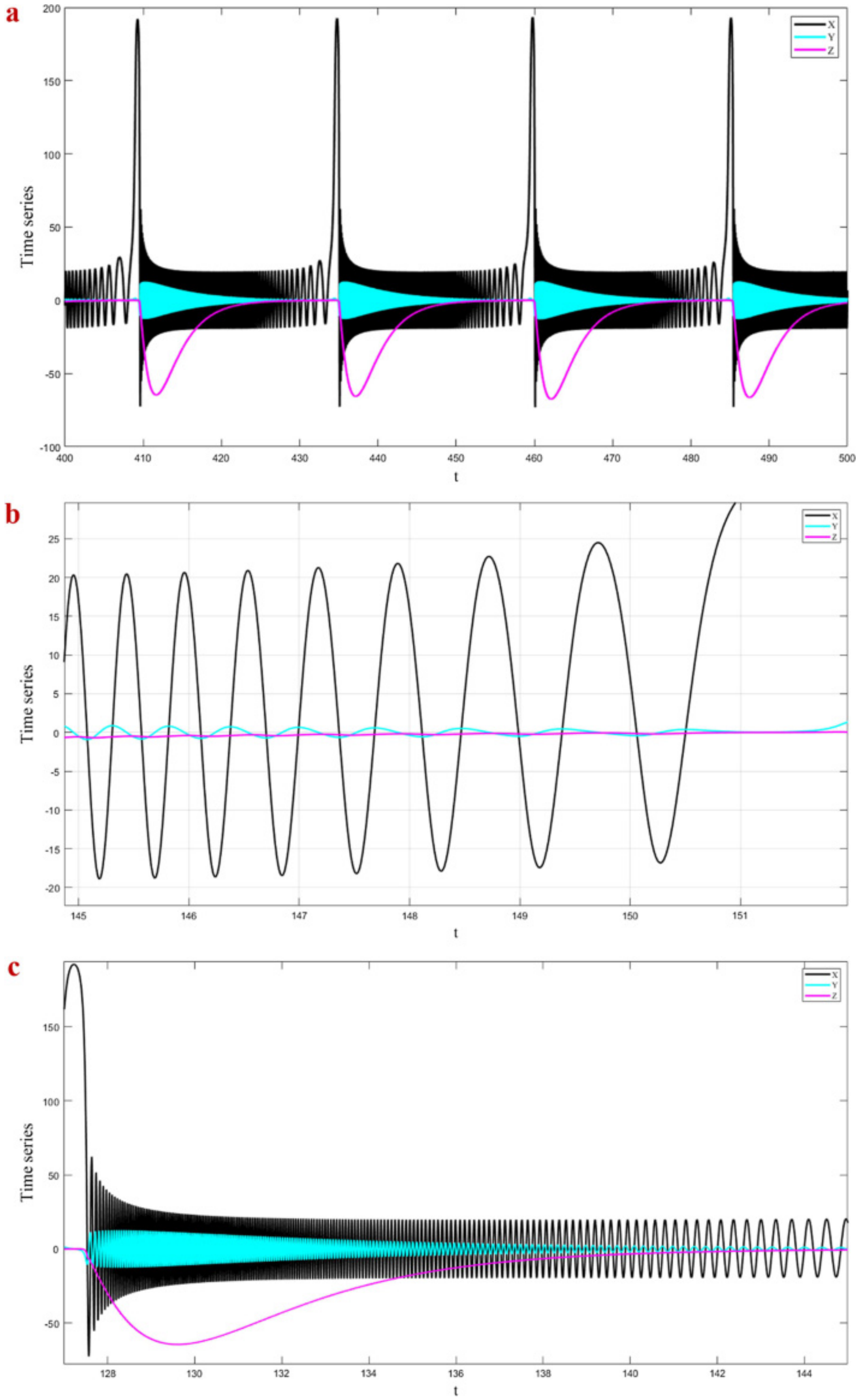

Fig. 3. (a) Three time series of the states X,Y, and Z variables for the chaotic attractor in black, cyan, and magenta color respectively; A zoomed view of time series in two parts of a cycle of variable Z; (b) in positive part; (c) in negative part.

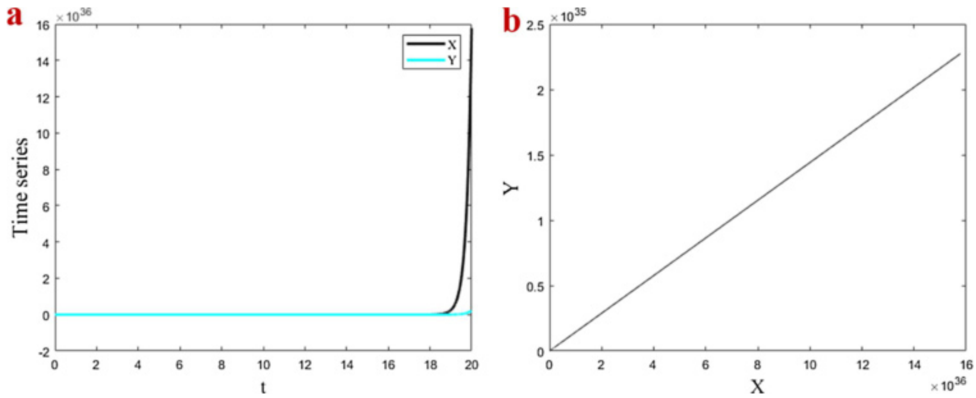

Fig. 4. (a) Time series and (b) state space of system (3) in $z^* = 0.06$.

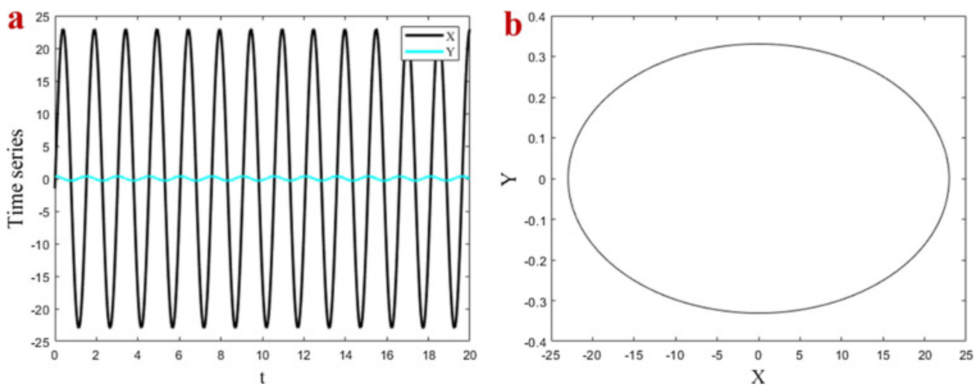

Fig. 5. (a) Time series and (b) state space of system (3) in $z^* = -0.06$.

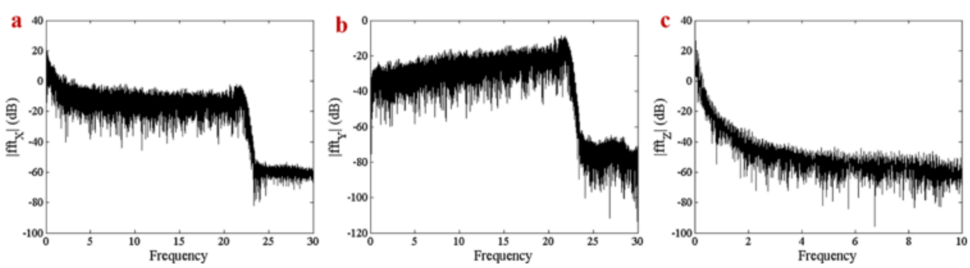

Fig. 6. The absolute value of FFT of (a) variable X; (b) variable Y; (c) variable Z in dB.

goes to infinity in this condition. Figure 5 depicts time series and state-space of the oscillatory dynamic of system (3) in $z^* = -0.06$.

The speed of variables can be compared using the fast Fourier transform (FFT) of the time series [26]. Figure 6 shows the absolute values of the FFT of variables. It can be seen that variables X and Y have power in much higher frequency than variable z.

To study the generation of chaotic attractors, the phase of variables is studied. Hilbert transform is used to extract the analytical signal of each variable, and then its phase is extracted as the spontaneous phase of each signal [27]. In part a of Figure 7

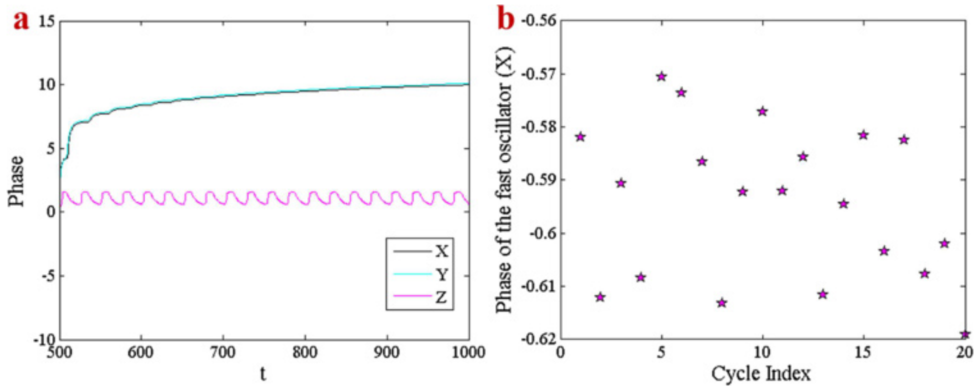

Fig. 7. (a) The logarithm of the spontaneous phase of X, Y and Z signals. (b) The phase of variable X in the zero cross of variable Z.

the logarithm of the spontaneous phase of x, y, and z signals is shown in black, cyan, and magenta color, respectively. The phase of fast variable x is extracted in the zero cross of slow variable z. Part b of Figure 7 shows the phase of fast variable x in the zero cross of variable z, which is calculated from the last 500 s of the time series of variable x when it is run for 1000 s. It is not clear where the chaos comes from, but it probably is related to the phase of the fast oscillator when the slow oscillator crosses zero. It can be seen that the phase does not have any regular pattern.

3.3 Bifurcation analysis of the proposed system

Dynamical properties of the system are investigated using the bifurcation diagram. Figure 8 shows Lyapunov exponents, Kaplan–Yorke dimension, and bifurcation diagram of system (1) for changing parameter a and constant initial conditions $(x_0, y_0, z_0) = (32, 0.1, 0)$. Before parameter $a = 218.4$, the variable x is unbounded. After parameter 218.4, a chaotic dynamic is generated which does not have any upper band for the parameter a. So in large values of parameter a, the dynamic of the system is chaotic and increasing the value of parameter a causes increase in the maximum peak of variable x. It appears there are many narrow periodic windows where the maximum Lyapunov exponent goes to zero.

4 Conclusion

In this paper, the simplest chaotic flow with a line of equilibria was proposed. Various studies have been done to understand the reason for generating the particular chaotic attractor of the proposed system. The stability of the line of equilibria was studied. The results showed that half of the line of equilibria was unstable while the other half was stable. Studying the chaotic attractor presented the switching rule of variable Z. This variable has switched the dynamic between an oscillator and an unstable dynamic. The FFT and Hilbert transform have been used to investigate the chaotic dynamic of the proposed system. Bifurcation analysis has shown that the proposed system had a chaotic dynamic for parameter a larger than a threshold.

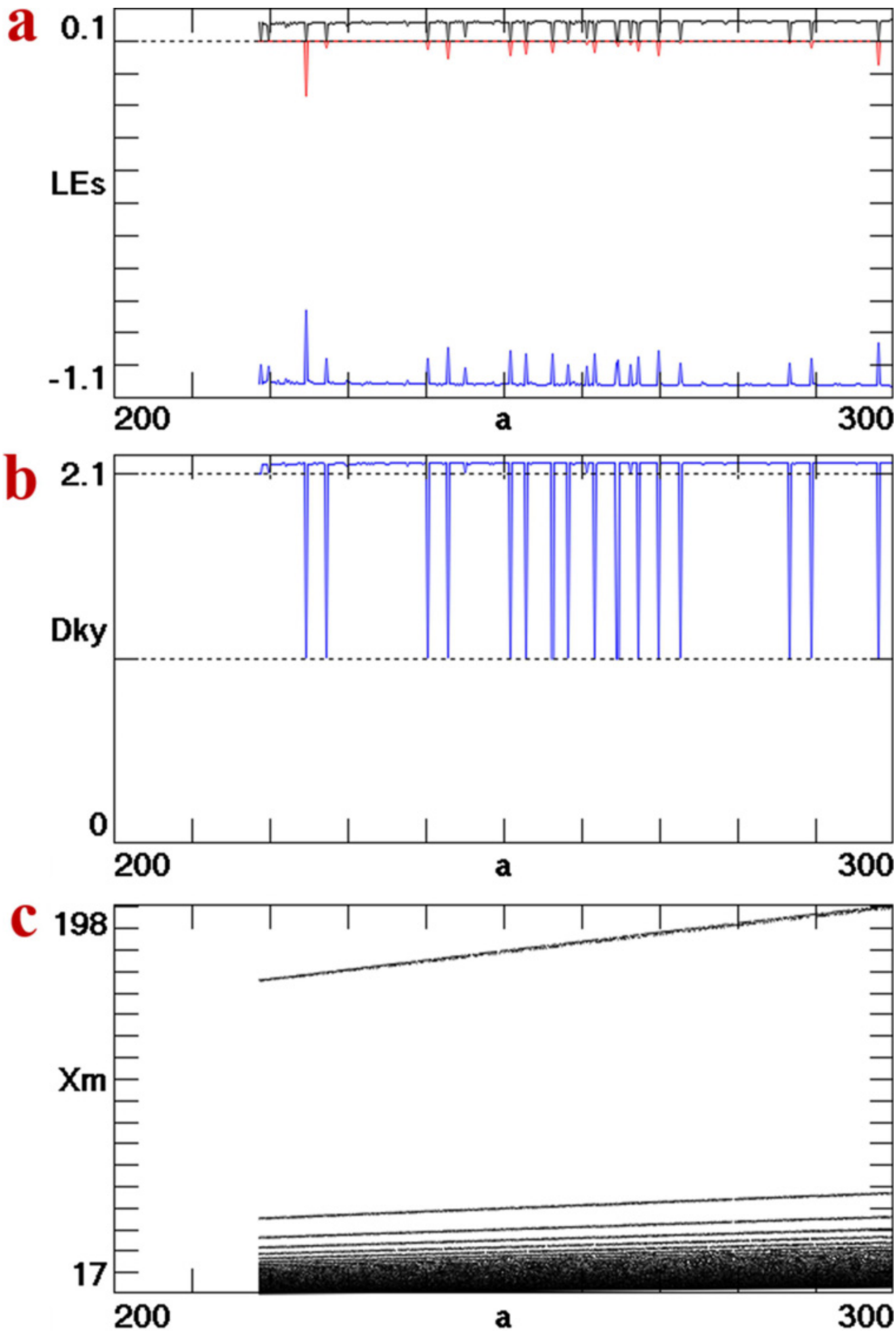

Fig. 8. (a) Lyapunov exponents, (b) Kaplan–Yorke dimension, and (c) bifurcation diagram of system (1) for changing parameter a and constant initial conditions $(x_0, y_0, z_0) = (32, 0.1, 0)$.

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