International Journal of Bifurcation and Chaos, Vol. 24, No. 10 (2014) 1450131 (7 pages)

DOI: 10.1142/S0218127414501314

Multistability in the Lorenz System: A Broken Butterfly

Chunbiao Li*,†,‡ and Julien Clinton Sprott†,§
*School of Electronic & Information Engineering,
Nanjing University of Information Science & Technology,
Nanjing 210044, P. R. China
*Engineering Technology Research and Development Center
of Jiangsu Circulation Modernization Sensor Network,
Jiangsu Institute of Commerce, Nanjing 210007, P. R. China
†Department of Physics, University of Wisconsin–Madison,
Madison, WI 53706, USA
‡chunbiaolee@gmail.com
§sprott@physics.wisc.edu

Received April 30, 2014

In this paper, the dynamical behavior of the Lorenz system is examined in a previously unexplored region of parameter space, in particular, where r is zero and b is negative. For certain values of the parameters, the classic butterfly attractor is broken into a symmetric pair of strange attractors, or it shrinks into a small attractor basin intermingled with the basins of a symmetric pair of limit cycles, which means that the system is bistable or tristable under certain conditions. Although the resulting system is no longer a plausible model of fluid convection, it may have application to other physical systems.

Keywords: Lorenz system; coexisting attractor; multistability.

1. Introduction

The Lorenz system [Lorenz, 1963]

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - xz - y, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

is one of the most widely studied of the many chaotic systems now known [Sprott, 2010], and it is the prototypical example of sensitive dependence on initial conditions (the butterfly effect). For the standard parameters used by Lorenz, $\sigma = 10$, $r = 28$, $b = 8/3$, there is a single symmetric double-wing chaotic attractor that resembles a butterfly. It is globally attracting, and the attractor is robust to relatively large variations of the parameters. The parameters were originally chosen to model atmospheric convection, but with a change in the values, the Lorenz equations have been used to model lasers [Roso-Franco & Corbalán, 1991], dynamos [Jones *et al.*, 1985], thermosyphons [Wu, 2011], waterwheels [Kolár & Gumbs, 1992], and chemical reactions [Poland, 1993]. Thus it is of interest to consider the entire region of parameter space, including negative and zero values of the parameters.

2. Lorenz System and the Dynamical Analysis of Its Subsets

2.1. *Basic properties of the Lorenz system*

The standard Lorenz system has seven terms, five of which are linear and two are quadratic. Since x, y, z, and t can be linearly rescaled, four of the seven coefficients can be set to ± 1, leaving exactly three parameters to completely characterize the system. Virtually all studies have assumed that these parameters are positive. One reason for this is that the system has attractors only if $\sigma + b + 1 > 0$, and it is globally attracting only if $\sigma > 0$ and $b > 0$ so that all three of the dynamic variables are damped. The energy required to maintain oscillations is provided by positive feedback. However, there are regimes for negative values of the parameters where periodic and chaotic solutions can occur, but they are not globally attracting. Table 1 lists the allowed dynamics in each of the eight octants of parameter space.

There is some connection between the Lorenz system and the Chen system [Lü *et al.*, 2002]. Moreover, there is a region in parameter space with $\sigma + r = -1$ where the Lorenz system is the time-reversed equivalent of the chaotic Chen system [Chen & Ueta, 1999], and thus there exists a strange repellor in that region [Algaba *et al.*, 2013a]. However, according to the conventional definitions of topological equivalence and conjugacy, they are nonequivalent because of the time reversal, which generally leads to a different flow orientation [Sprott *et al.*, 2014].

Normally a positive r is required to provide the energy through positive feedback necessary to keep oscillations from damping. However, with $b < 0$, that energy can be provided through the anti-damping in the $\dot{z}$ equation. Since chaos can occur for both positive and negative r with $b < 0$, we focus on the case with $r = 0$ because it has features common to the two regimes and it reduces the parameter space to two dimensions, allowing it to be completely explored. In this case, the time-reversed Chen system corresponds to $\sigma = -1$.

Table 1. Allowed dynamics in the octants of $\sigma r b$ space.

(σ, r, b)	Dynamics			
$(+, +, +)$	S	P	C	
$(+, +, -)$		P	C	U
$(+, -, +)$	S			
$(+, -, -)$	S	P	C	U
$(-, +, +)$	S	P	C	U
$(-, +, -)$				U
$(-, -, +)$				U
$(-, -, -)$				U

Note: S = stable equilibrium, P = periodic, C = chaotic, U = unbounded.

We now give the basic features of the Lorenz system for $r = 0$. Firstly, the system has rotational symmetry about the z-axis. All equilibrium points are also symmetric with respect to the z-axis. Secondly, the system is dissipative when $b + \sigma > -1$. Finally, the system retains its three real equilibria at $P_1 = (0, 0, 0)$, $P_2 = (\sqrt{-b}, \sqrt{-b}, -1)$, and $P_3 = (-\sqrt{-b}, -\sqrt{-b}, -1)$.

The eigenvalues of the Jacobian matrix at P_1 are $\lambda_1 = -\sigma$, $\lambda_2 = -1$, and $\lambda_3 = -b$. For $b < 0$, λ_3 is always positive, and thus the origin is an unstable saddle-node. The other two equilibrium points, P_2 and P_3, are saddle-foci, whose eigenvalues contain one negative real number, and a pair of complex conjugate eigenvalues with positive real parts. Thus system (1) with $r = 0$ and $b < 0$ is unstable at all three equilibrium points.

When time is reversed, all orbits are unbounded for all initial conditions, which means that the only repellor for $r = 0$ and $b < 0$ is the one at infinity.

When $b = 0$, the system (1) has an infinite line of equilibrium points $(0, 0, z)$ whose eigenvalues are $(0, -0.5(\sigma + 1) \pm 0.5\sqrt{(\sigma - 1)^2 - 4z\sigma + 4r\sigma})$. The line is neutrally stable for $z > r - 1$ and unstable for $z < r - 1$ at all positive values of σ.

2.2. *Dynamical regions for $r = 0$ and negative b*

There is equivalent of a strange attractor in Lorenz system to the time-reversed Lü system when $r = 0$ [Algaba *et al.*, 2013b]. The dynamical regions for system (1) with $r = 0$ in σb space are shown in Fig. 1.

(1) The region in the rectangle $b \in (0, -1)$ and $\sigma \in (0, 0.3)$ satisfies $\sigma + b > -1$, and thus the system is dissipative.
(2) The majority of solutions are unbounded, with many separate regions of periodic and chaotic solutions.
(3) The main region of bounded solutions is in the vicinity of the line $b = 2.17\sigma - 0.866$.
(4) In the upper-left corner of Fig. 1, unbounded bands separate a fractal series of limit cycles and chaos.
(5) The green region at small σ and the red region at small b are numerical artifacts that come about because the orbit remains for long times near the equilibrium point P_1 at the origin.

To better illustrate the dynamics in the large bounded region of Fig. 1, a plot of the Lyapunov

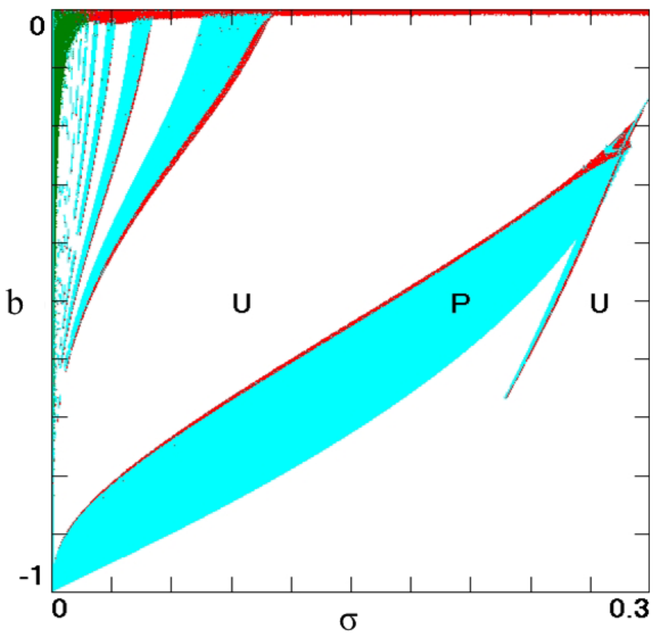
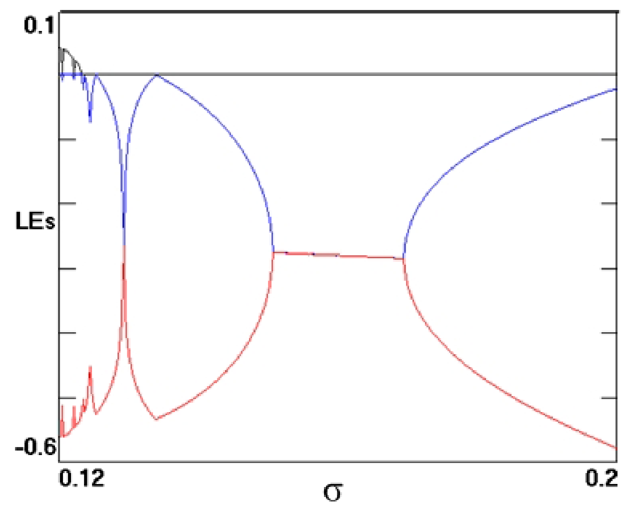
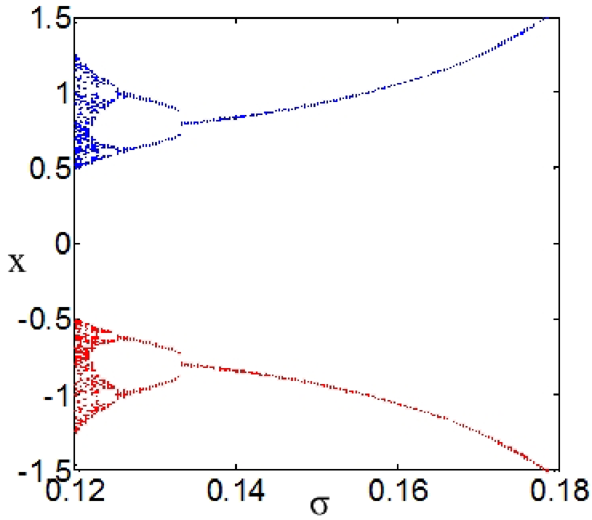


Fig. 1. Regions of various dynamical behaviors as a function of the bifurcation parameters σ and b with $r = 0$. The chaotic regions are shown in red, the periodic regions are shown in light blue, and the unbounded regions are shown in white.



(a)



(b)

Fig. 2. Lyapunov exponent spectrum and bifurcation diagram for symmetric initial conditions (red for $(-0.8, 3, 0)$, and blue for $(0.8, -3, 0)$) showing a period-doubling route to chaos.

exponents and the values of x when $z = 0$ for $b = -0.6$ and σ in the range [0.12, 0.2] are shown in Fig. 2. As σ decreases, a pair of limit cycles undergoes period doubling, forming a pair of strange attractors that approach the equilibrium at the origin and are destroyed in a boundary crisis at $\sigma \approx 0.12$.

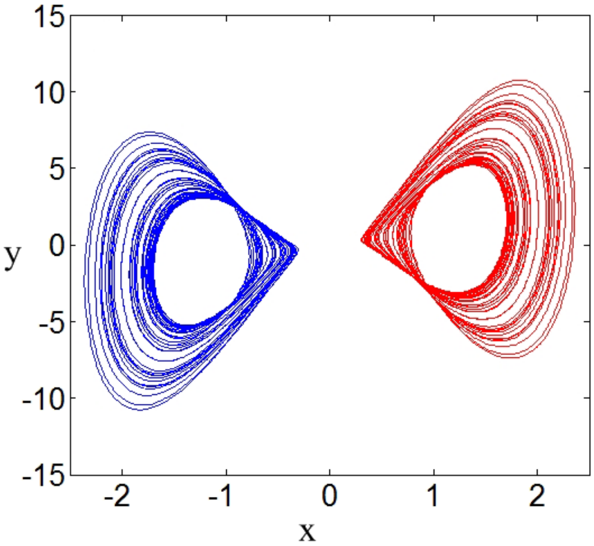
2.3. *Broken butterfly*

Figure 2 shows that when $b = -0.6$, there is a range of σ where two strange attractors coexist. For $\sigma = 0.12$, the attractors form a broken butterfly as shown in Fig. 3. The calculated Lyapunov exponents are $L_1 = 0.0413$, $L_2 = 0$, and $L_3 = -0.5614$, and the Kaplan–Yorke dimension is $D_{KY} = 2 - L_1/L_3 = 2.0736$. Note that the variable z is both positive and negative in contrast to the classic Lorenz system where it is always positive.

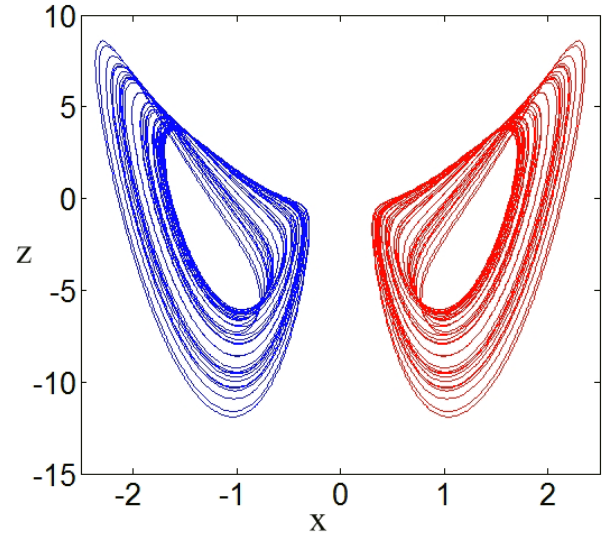
The equilibrium points are $P_1 = (0, 0, 0)$, $P_2 = (\sqrt{0.6}, \sqrt{0.6}, -1)$, and $P_3 = (-\sqrt{0.6}, -\sqrt{0.6}, -1)$. The eigenvalues of P_1 are $-0.12, -1, 0.6$, and the eigenvalues for P_2 and P_3 are $\lambda_1 = -0.8212$, $\lambda_2 = 0.1506 + 0.3907i$, and $\lambda_3 = 0.1506 - 0.3907i$. Thus the origin is a saddle-node, and the other equilibria are saddle-foci.

3. Multistablity with Coexisting Attractors

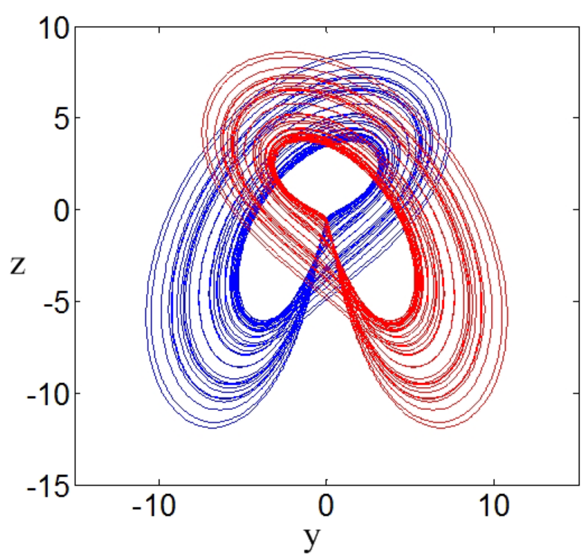
Hidden in the region of negative b and positive σ are other examples of multistablity. Besides the bistablity with a broken butterfly attractor, there are other bistablities including symmetric pairs of limit cycles. Furthermore, for a small range of parameters, there are three coexisting attractors, two of which are a symmetric pair of limit cycles, and the third is a symmetric limit cycle or strange attractor.



(a)

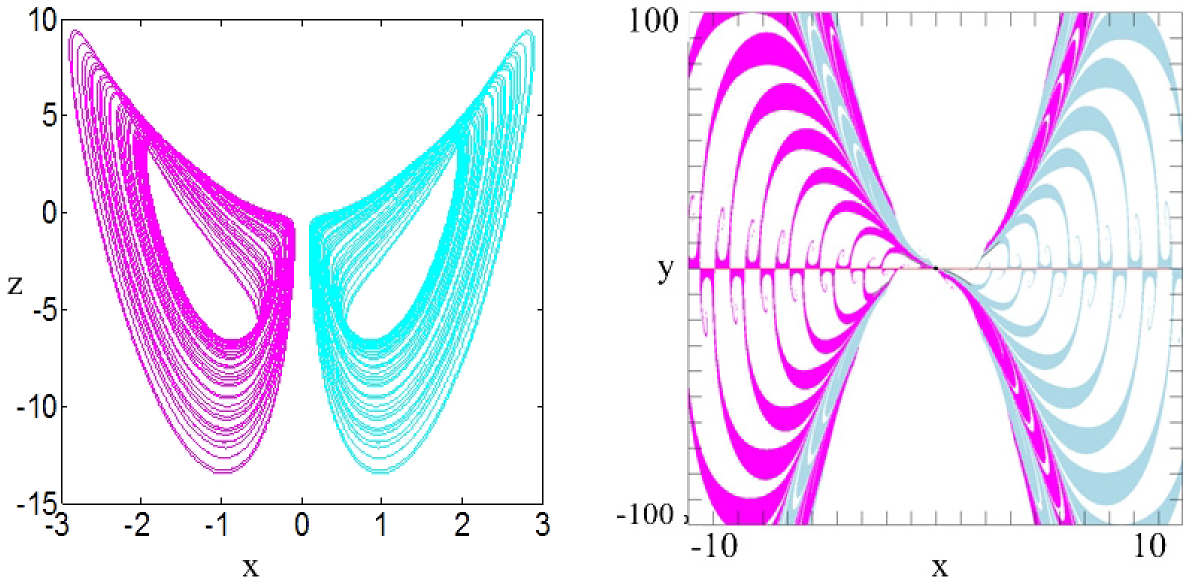


(b)



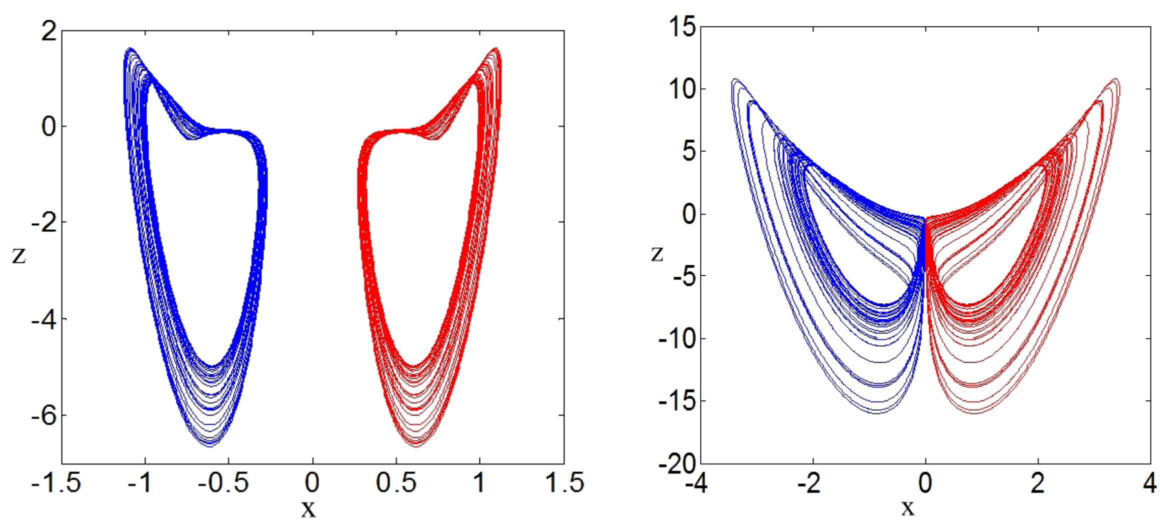
(c)

Fig. 3. Broken butterfly (blue and red attractors correspond to two symmetric initial conditions $(\mp 0.8, \pm 3, 0)$). (a) x–y plane, (b) x–z plane and (c) y–z plane.



(a) (b)

Fig. 4. Coexisting strange attractors and their fractal basins of attraction. (a) Strange attractors with initial conditions $(\pm 0.8, \mp 3, 0)$ and (b) cross-section for $z = 0$ of the basins of attraction.



(a) (b)

Fig. 5. Attractors for $r = 0$, $b = -0.3$ (blue and red attractors correspond to two symmetric initial conditions, and the third coexisting attractor is in black). (a) $\sigma = 0.052$ with initial conditions $(\mp 1, 0, 1)$, (b) $\sigma = 0.256$ with initial conditions $(\mp 0.1, \pm 0.1, -2)$, (c) $\sigma = 0.265$ with initial conditions $(\mp 0.1, \pm 0.1, -2)$, (d) $\sigma = 0.272$ with initial conditions $(\mp 0.1, \pm 0.1, -2)$, (e) $\sigma = 0.277$ with initial conditions $(-0.1, 0.1, -2)$ and $(\mp 0.1, \pm 0.1, -13)$ and (f) $\sigma = 0.279$ with initial conditions $(-0.1, 0.1, -2)$ and $(\mp 0.1, \pm 0.1, -14)$.

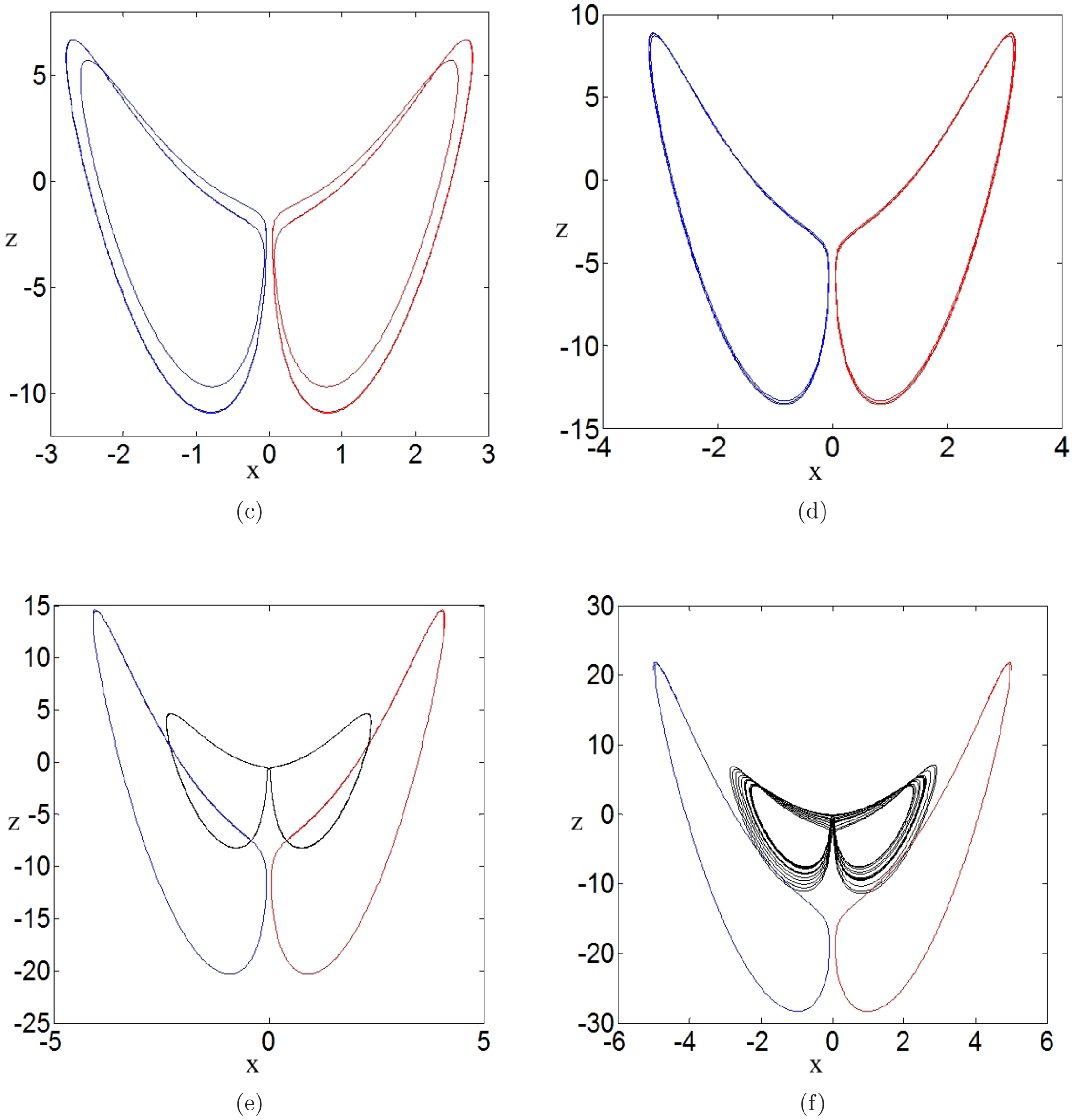


Fig. 5. (*Continued*)

For example, when $r = 0$, $\sigma = 0.192$, $b = -0.45$, a symmetric pair of strange attractors coexist, as shown in Fig. 4(a) with basins of attraction as shown in Fig. 4(b). The basins of attraction for the two chaotic attractors are indicated by pink and light blue, respectively. The basins have the expected symmetry about the z-axis and an intricate fractal structure. Most initial conditions lead to unbounded orbits as indicated by white.

When $r = 0$, $b = -0.3$, and σ varying, many coexisting attractors exist. Figure 5(a) shows a typical symmetric pair of strange attractors in the region where σ is relatively small. As σ increases, these attractors are destroyed, and there is a large region of unbounded solutions until a new pair of almost touching strange attractors appears as in Fig. 5(b). These attractors undergo an inverse period doubling as shown in Figs. 5(c) and 5(d), forming a pair of nearly touching limit cycles. Then a third symmetric limit cycle is born as shown in Fig. 5(e). With a further increase of σ, the symmetric limit cycle changes into a strange attractor that coexists with the symmetric pair of limit cycles as shown in Fig. 5(f). The basins of attraction of the three coexisting attractors are shown in Fig. 6.

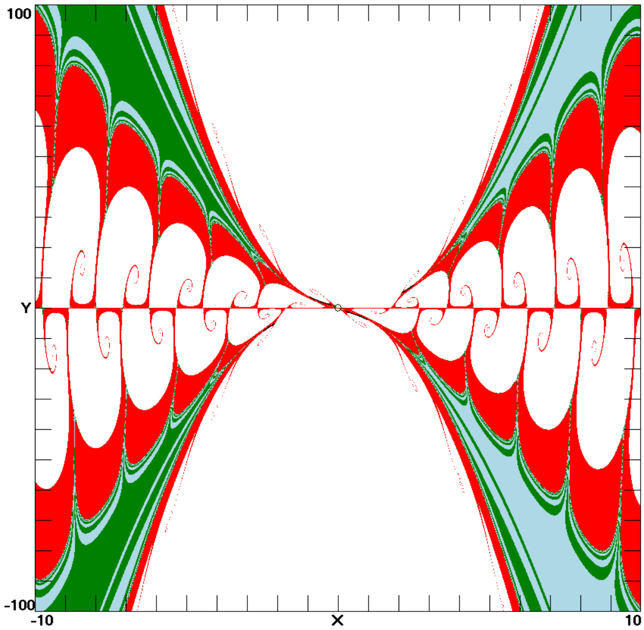


Fig. 6. Cross-section for $z = 0$ of the basins of attraction for the symmetric pair of limit cycles (light blue and green) and the strange attractor (red) of system (1) at $r = 0$, $b = -0.3$, $\sigma = 0.279$.

4. Conclusions and Discussion

In this unexplored and somewhat unphysical regime of the Lorenz system, new regions of chaos and multistability have been found. The chaotic solutions occur in a very small region of parameter space and with small basins of attraction. The region where a strange attractor coexists with limit cycles is a small subset of these regions, which probably accounts for it not having been previously reported. It remains to be shown to what extent the Lorenz system with $r = 0$ and negative b is of physical relevance, but it also indicates that there is more to learn even with this old and extensively studied system.

Acknowledgments

We thank John Young for helpful discussions and suggestions. This work was also supported financially by the Jiangsu Overseas Research & Training Program for University Prominent Young and Middle-aged Teachers and Presidents, the 4th 333 High-level Personnel Training Project (Su Talent [2011] No. 15) and the National Science Foundation for Postdoctoral General Program and Special Founding Program of People's Republic of China (Grant Nos. 2011M500838 and 2012T50456) and Postdoctoral Research Foundation of Jiangsu Province (Grant No. 1002004C).

References

Algaba, A., Fernández-Sánchez, F., Merino, M. & Rodríguez-Luis, A. J. [2013a] "Chen's attractor exists if Lorenz repulsor exists: The Chen system is a special case of the Lorenz system," *Chaos* **23**, 033108.

Algaba, A., Fernández-Sánchez, F., Merino, M. & Rodríguez-Luis, A. J. [2013b] "The Lü system is a particular case of the Lorenz system," *Phys. Lett. A* **377**, 2771–2776.

Chen, G. & Ueta, T. [1999] "Yet another chaotic attractor," *Int. J. Bifurcation and Chaos* **9**, 1465–1466.

Jones, C. A., Weiss, N. O. & Cattaneo, F. [1985] "Nonlinear dynamos: A complex generalization of the Lorenz equations," *Physica D* **14**, 161–176.

Kolár, M. & Gumbs, G. [1992] "Theory for the experimental observation of chaos in a rotating waterwheel," *Phys. Rev. A* **45**, 626–637.

Lorenz, E. N. [1963] "Deterministic nonperiodic flow," *J. Atmos. Sci.* **20**, 130–141.

Lü, J., Chen, G. & Zhang, S. [2002] "The compound structure of a new chaotic attractor," *Chaos Solit. Fract.* **14**, 669–672.

Poland, D. [1993] "Cooperative catalysis and chemical chaos: A chemical model for the Lorenz equations," *Physica D* **65**, 86–99.

Roso-Franco, L. & Corbalán, R. [1991] "Generalization of the Lorenz model to study the dynamics of a laser operating at two different transverse electromagnetic modes," *Europhys. Lett.* **16**, 183–188.

Sprott, J. C. [2010] *Elegant Chaos: Algebraically Simple Chaotic Flows* (World Scientific, Singapore).

Sprott, J. C., Wang, X. & Chen, G. [2014] "When two dual chaotic systems shake hands," *Int. J. Bifurcation and Chaos* **24**, 1450086-1–3.

Wu, Y. [2011] "On the modeling and control of coupled multi-loop thermosyphons," *Proc. 2011 American Conf. Applied Mathematics and the 5th WSEAS Int. Conf. Computer Engineering and Applications*, pp. 105–110.