About structural stability of 3-D quadratic mappings

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Abstract

In this paper, we discuss the concept of structural stability as a criterion for robustness of invariant sets in 3-D quadratic mappings. We give the exact form of the small perturbation for these maps. The relevance of this result is that the most results in the literature do not give any form for these perturbations.

Keywords: Structural stability, 3-D quadratic map, chaos
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1 Introduction

The most general 3-D quadratic map is given by

$$f : \begin{cases} x' = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz \\ y' = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5y^2 + b_6z^2 + b_7xy + b_8xz + b_9yz \\ z' = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 + c_6z^2 + c_7xy + c_8xz + c_9yz \end{cases} \quad (1)$$

where $(a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ are the bifurcation parameters. Maps of the form (1) are part of the models of storage ring elements in the "thin lens" approximation as shown in [21]. Several phenomena occur in some 3-D quadratic maps of the form (1) such as hyperchaotic and wild-hyperbolic attractors as shown in [3-5-7-8-9-10-11]. Many of these maps are useful in potential applications [4-6]. Some generalizations of the 2-D Hénon map were introduced in [7-8-9-10-11], and the attractors obtained are very similar to the Lorenz and Shimizu-Morioka attractors as shown in [1-2-3-7]. In particular, these special cases are used in the study of homoclinic phenomena and the unfolding of 2-D maps to maps of higher dimension [8-9].

In this paper, we discuss the concept of structural stability as a criterion for robustness of invariant sets of the general 3-D quadratic map. We define and state important properties of this notion, along with its conditions. We will give the exact form of the C^r small perturbation (see the definition below) for the general 3-D quadratic map (1). The relevance of this result is that almost all results in the literature do not give any form for these perturbations.

2 Structural stability of 3-D quadratic mappings

The concept of structural stability was introduced by Andronov and Pontryagin in 1937, and it plays an important role in the theory of dynamical systems. Conditions for structural stability of high-dimensional systems were formulated in [23] as follows: A system must satisfy both **Axiom A** and the strong transversality condition. From a mathematical point of view, let $C^r(\mathbb{R}^n, \mathbb{R}^n)$ denote the space of C^r vector fields of $\mathbb{R}^n$ into $\mathbb{R}^n$. Let $Diff^r(\mathbb{R}^n, \mathbb{R}^n)$ be the subset of $C^r(\mathbb{R}^n, \mathbb{R}^n)$ consisting of the C^r diffeomorphisms: (a) Two elements of $C^r(\mathbb{R}^n, \mathbb{R}^n)$ are C^r ε-close $(k \leq r)$, or just C^k close, if they, along with their first k derivatives, are within ε as measured in some norm. (b) A dynamical system (vector field or map) is structurally stable if nearby systems have the same qualitative dynamics.

The words *nearby systems* can be translated in terms of C^k conjugate for maps and C^k equivalence for vector fields. Assume that the maps under consideration act on compact, boundaryless, n-dimensional, differentiable manifolds M, rather than all of $\mathbb{R}^n$. This assumption induces the so-called C^k

topology given in [24]: The C^k topology is the topology induced on $C^r(M, M)$ by the measure of distance between two elements of $C^r(M, M)$. Hence, the notion of structural stability can be formally formulated as follows: Consider a map $f \in Diff^r(M, M)$ (resp. a C^r vector field in $C^r(M, M)$). Then f is structurally stable if there exists a neighborhood N of f in the C^k topology such that f is C^0 conjugate (resp. C^0 equivalent) to every map (resp. vector field) in N. From this definition, it follows that structural stability implies a common and typical or *generic* property of a dynamical system to a dense set of dynamical systems in $C^r(M, M)$. We note that a property of a map (resp. vector field) is said to be C^k generic if the set of maps (resp. vector fields) possessing that property contains a residual subset in the C^k topology.

It was proved in [24] that the hyperbolic fixed points, periodic orbits, and the transversal intersection of the stable and unstable manifolds of hyperbolic fixed points and periodic orbits are structurally stable and generic. Also, structurally stable systems are generic. In terms of C^k topology, the structural stability can be reformulated as follows: A diffeomorphism f is C^r structurally stable if, for any C^r small perturbation g of f, there is a homeomorphism h of the phase space such that

$$h \circ f\,(x, y, z) = g \circ h\,(x, y, z) \tag{2}$$

for all points x in the phase space.

Generally, the determination of a C^r small perturbation g of f is not an easy task. Most results in the literature do not give any form of g. In this paper, we will give the exact form of the C^r small perturbation g for the general 3-D quadratic map (1). Indeed, assume that the C^r small perturbation g of f is also a 3-D quadratic map of the form

$$g : \begin{cases} x' = d_0 + d_1x + d_2y + d_3z + d_4x^2 + d_5y^2 + d_6z^2 + d_7xy + d_8xz + d_9yz \\ y' = e_0 + e_1x + e_2y + e_3z + e_4x^2 + e_5y^2 + e_6z^2 + e_7xy + e_8xz + e_9yz \\ z' = f_0 + f_1x + f_2y + f_3z + f_4x^2 + f_5y^2 + f_6z^2 + f_7xy + f_8xz + f_9yz \end{cases} \tag{3}$$

where $(d_i, e_i, f_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ are small perturbations of $(a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ to be determined if we assume that f and g are topologically conjugate. The linear transformation h is defined by

$$h\,(x, y, z) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4}$$

with the condition of invertibility given by

$$d = (h_{22}h_{33} - h_{23}h_{32})\,h_{11} + (h_{31}h_{23} - h_{21}h_{33})\,h_{12} + (h_{21}h_{32} - h_{22}h_{31})\,h_{13} \neq 0. \tag{5}$$

The simplest assumption is that h is linear. Therefore, all the results related to the existence of this transformation are part of the whole set of possible existing transformations.

Note that if such a transformation h exists, then there is an equivalence relation, and the set of all maps is divided into classes of topologically conjugate maps. This implies that f and g have identical topological properties. In particular, they have the same number of fixed and periodic points of the same stability types. If f and g are invertible, the order of the points is preserved, and if the maps are non-invertible, the order of points is also preserved, but h maps forward orbits of f onto the corresponding forward orbits of g.

The direct application of (2) requires tedious and very complicated calculations. To avoid this problem, we use some results from linear algebra. Indeed, it is well known that any matrix over $\mathbb{R}$ is similar to an upper triangular matrix J which is the Jordan normal form. Finding this form is related to knowing of the minimal polynomial. For 3×3 matrices, we have 6 cases of Jordan normal forms as shown in [22]:

$$\begin{cases} J_1 = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 0 \\ 0 & 0 & h_{33} \end{pmatrix}, J_2 = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 0 \\ 0 & 0 & h_{22} \end{pmatrix} \\ J_3 = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 1 \\ 0 & 0 & h_{22} \end{pmatrix}, J_4 = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & h_{11} \end{pmatrix} \\ J_5 = \begin{pmatrix} h_{11} & 1 & 0 \\ 0 & h_{11} & 0 \\ 0 & 0 & h_{11} \end{pmatrix}, J_6 = \begin{pmatrix} h_{11} & 1 & 0 \\ 0 & h_{11} & 1 \\ 0 & 0 & h_{11} \end{pmatrix}. \end{cases} \tag{6}$$

Here $h_{ii} \neq 0$ due to condition (5). Assume that the matrix defining the transformation h is one of the matrices $(J_i)_{1 \leq i \leq 6}$. Then the formula for h is well defined, and we must look for the possible values of the parameters $(d_i, e_i, f_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ in terms of $(a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ and $(h_{ii})_{1 \leq i \leq 3}$ defining the matrices $(J_i)_{1 \leq i \leq 6}$. Indeed, for the first case (J_1) we have

$$
\begin{cases}
d_0 = a_0 h_{11}, d_1 = a_1 \frac{h_{11}}{h_{22}}, d_2 = a_2, d_3 = a_3 \frac{h_{11}}{h_{33}}, d_4 = \frac{a_4}{h_{11}}, d_5 = a_5 \frac{h_{11}}{h_{22}^2} \\
d_6 = a_6 \frac{h_{11}}{h_{33}^2}, d_7 = \frac{a_7}{h_{22}}, d_8 = \frac{a_8}{h_{33}}, d_9 = a_9 \frac{h_{11}}{h_{22} h_{33}} \\
e_0 = b_0 h_{22}, e_1 = \frac{b_1}{h_{11}} h_{22}, e_2 = b_2, e_3 = b_3 \frac{h_{22}}{h_{33}}, e_4 = \frac{b_4}{h_{11}^2} h_{22}, \\
e_5 = \frac{b_5}{h_{22}}, e_6 = b_6 \frac{h_{22}}{h_{33}^2}, e_7 = \frac{b_7}{h_{11}}, e_8 = \frac{b_8}{h_{11}} \frac{h_{22}}{h_{33}}, e_9 = \frac{b_9}{h_{33}} \\
f_0 = c_0 h_{33}, f_1 = \frac{c_1}{h_{11}} h_{33}, f_2 = \frac{c_2}{h_{22}} h_{33}, f_3 = c_3, f_4 = \frac{c_4}{h_{11}^2} h_{33} \\
f_5 = \frac{c_5}{h_{22}^2} h_{33}, f_6 = \frac{c_6}{h_{33}}, f_7 = \frac{c_7}{h_{11} h_{22}} h_{33}, f_8 = \frac{c_8}{h_{11}}, f_9 = \frac{c_9}{h_{22}}.
\end{cases} \tag{7}
$$

For the second case (J_2) we have

$$
\begin{cases}
d_0 = a_0 h_{11}, d_1 = a_1, d_2 = \frac{a_2 h_{11}}{h_{22}}, d_3 = \frac{a_3 h_{11}}{h_{22}}, d_4 = \frac{a_4}{h_{11}} \\
d_5 = \frac{a_5 h_{11}}{h_{22}^2}, d_6 = \frac{a_6 h_{11}}{h_{22}^2}, d_7 = \frac{a_7}{h_{22}}, d_8 = \frac{a_8}{h_{22}}, d_9 = \frac{a_9 h_{11}}{h_{22}^2} \\
e_0 = b_0 h_{22}, e_1 = \frac{b_1 h_{22}}{h_{11}}, e_2 = b_2, e_3 = b_3, e_4 = \frac{b_4 h_{22}}{h_{11}^2} \\
e_5 = \frac{b_5}{h_{22}}, e_6 = \frac{b_6}{h_{22}}, e_7 = \frac{b_7}{h_{11}}, e_8 = \frac{b_8}{h_{11}}, e_9 = \frac{b_9}{h_{22}} \\
f_0 = c_0 h_{22}, f_1 = \frac{c_1 h_{22}}{h_{11}}, f_2 = c_2, f_3 = c_3, f_4 = \frac{c_4 h_{22}}{h_{11}^2} \\
f_5 = \frac{c_5 h_{22}}{h_{22}^2}, f_6 = \frac{c_6}{h_{22}}, f_7 = \frac{c_7}{h_{11}}, f_8 = \frac{c_8}{h_{11}}, f_9 = \frac{c_9}{h_{22}}.
\end{cases} \tag{8}
$$

For the third case (J_3) we have

$$
\begin{cases}
d_0 = a_0 h_{11}, d_1 = a_1, d_2 = \frac{a_2 h_{11}}{h_{22}}, d_3 = -h_{11} \frac{a_2 - a_3 h_{22}}{h_{22}^2}, d_4 = \frac{a_4}{h_{11}}, d_5 = \frac{a_5 h_{11}}{h_{22}^2} \\
d_6 = h_{11} \frac{a_5 - a_9 h_{22} + a_6 h_{22}^2}{h_{22}^4}, d_7 = \frac{a_7}{h_{22}}, d_8 = -\frac{a_7 - a_8 h_{22}}{h_{22}^2}, d_9 = -h_{11} \frac{2a_5 - a_9 h_{22}}{h_{22}^3} \\
e_0 = c_0 + b_0 h_{22}, e_1 = \frac{c_1 + b_1 h_{22}}{h_{11}}, e_2 = \frac{c_2 + b_2 h_{22}}{h_{22}}, e_3 = -\frac{c_2 + (b_2 - c_3) h_{22} - b_3 h_{22}^2}{h_{22}^2} \\
e_4 = \frac{c_4 + b_4 h_{22}}{h_{11}^2}, e_5 = \frac{b_5 h_{22} + c_5}{h_{22}^2}, e_6 = \frac{(b_5 - c_9) h_{22} + (c_6 - b_9) h_{22}^2 + b_6 h_{22}^3 c_5}{h_{22}^4} \\
e_7 = \frac{c_7 + b_7 h_{22}}{h_{11} h_{22}}, e_8 = -\frac{c_7 + (b_7 - c_8) h_{22} - b_8 h_{22}^2}{h_{11} h_{22}^2}, e_9 = -\frac{2c_5 + (2b_5 - c_9) h_{22} - b_9 h_{22}^2}{h_{22}^3} \\
f_0 = c_0 h_{22}, f_1 = \frac{c_1 h_{22}}{h_{11}}, f_2 = c_2, f_3 = -\frac{c_2 - c_3 h_{22}}{h_{22}}, f_4 = \frac{c_4 h_{22}}{h_{11}^2} \\
f_5 = \frac{c_5}{h_{22}}, f_6 = \frac{c_5 - c_9 h_{22} + c_6 h_{22}^2}{h_{22}^3}, f_7 = \frac{c_7}{h_{11}}, f_8 = -\frac{c_7 - c_8 h_{22}}{h_{11} h_{22}}, f_9 = -\frac{2c_5 - c_9 h_{22}}{h_{22}^2}.
\end{cases} \tag{9}
$$

For the fourth case (J_4) we have

$$
\begin{cases}
d_0 = a_0 h_{11}, d_1 = a_1, d_2 = a_2, d_3 = a_3, d_4 = \frac{a_4}{h_{11}}, d_5 = \frac{a_5}{h_{11}} \\
d_6 = \frac{a_6}{h_{11}}, d_7 = \frac{a_7}{h_{11}}, d_8 = \frac{a_8}{h_{11}}, d_9 = \frac{a_9}{h_{11}} \\
e_0 = b_0 h_{11}, e_1 = b_1, e_2 = b_2, e_3 = b_3, e_4 = \frac{b_4}{h_{11}}, e_5 = \frac{b_5}{h_{11}} \\
e_6 = \frac{b_6}{h_{11}}, e_7 = \frac{b_7}{h_{11}}, e_8 = \frac{b_8}{h_{11}}, e_9 = \frac{b_9}{h_{11}} \\
f_0 = c_0 h_{11}, f_1 = c_1, f_2 = c_2, f_3 = c_3, f_4 = \frac{c_4}{h_{11}}, f_5 = \frac{c_5}{h_{11}} \\
f_6 = \frac{c_6}{h_{11}}, f_7 = \frac{c_7}{h_{11}}, f_8 = \frac{c_8}{h_{11}}, f_9 = \frac{c_9}{h_{11}}.
\end{cases}
\tag{10}
$$

For the fifth case (J_5) we have

$$
\begin{cases}
d_0 = b_0 + a_0 h_{11}, d_1 = \frac{b_1 + a_1 h_{11}}{h_{11}}, d_2 = -\frac{b_1 + (a_1 - b_2) h_{11} - a_2 h_{11}^2}{h_{11}^2} \\
d_3 = \frac{b_3 + a_3 h_{11}}{h_{11}}, d_4 = \frac{b_4 + a_4 h_{11}}{h_{11}^2}, d_5 = \frac{b_4 + (a_4 - b_7) h_{11} + a_5 h_{11}^3 + (b_5 - a_7) h_{11}^2}{h_{11}^4} \\
d_6 = \frac{b_6 + a_6 h_{11}}{h_{11}^2}, d_7 = -\frac{2b_4 + (2a_4 - b_7) h_{11} - a_7 h_{11}^2}{h_{11}^3} \\
d_8 = \frac{b_8 + a_8 h_{11}}{h_{11}^2}, d_9 = -\frac{b_8 + (a_8 - b_9) h_{11} - a_9 h_{11}^2}{h_{11}^3} \\
e_0 = b_0 h_{11}, e_1 = b_1, e_2 = -\frac{b_1 - b_2 h_{11}}{h_{11}}, e_3 = b_3, e_4 = \frac{b_4}{h_{11}} \\
e_5 = \frac{b_4 - b_7 h_{11} + b_5 h_{11}^2}{h_{11}^3}, e_6 = \frac{b_6}{h_{11}}, e_7 = -\frac{2b_4 - b_7 h_{11}}{h_{11}^2}, e_8 = \frac{b_8}{h_{11}}, e_9 = -\frac{b_8 - b_9 h_{11}}{h_{11}^2} \\
f_0 = c_0 h_{11}, f_1 = c_1, f_2 = -\frac{c_1 - c_2 h_{11}}{h_{11}}, f_3 = c_3, f_4 = \frac{c_4}{h_{11}} \\
f_5 = \frac{c_4 - c_7 h_{11} + c_5 h_{11}^2}{h_{11}^3}, f_6 = \frac{c_6}{h_{11}}, f_7 = -\frac{2c_4 - c_7 h_{11}}{h_{11}^2}, f_8 = \frac{c_8}{h_{11}}, f_9 = -\frac{c_8 - c_9 h_{11}}{h_{11}^2}.
\end{cases}
\tag{11}
$$

For the sixth case (J_6) we have

$$
\begin{cases}
d_0 = b_0 + a_0 h_{11}, d_1 = \frac{b_1 + a_1 h_{11}}{h_{11}}, d_2 = -\frac{b_1 - a_2 h_{11}^2 + (a_1 - b_2) h_{11}}{h_{11}^2} \\
d_3 = \frac{b_1 + (a_1 - b_2) h_{11} + (b_3 - a_2) h_{11}^2 + a_3 h_{11}^3}{h_{11}^3} \\
d_4 = \frac{b_4 + a_4 h_{11}}{h_{11}^2}, d_5 = \frac{b_4 + (a_4 - b_7) h_{11} + (b_5 - a_7) h_{11}^2 + a_5 h_{11}^3}{h_{11}^4} \\
d_6 = \frac{b_4 + (a_4 - b_7) h_{11} + (b_8 - a_7 + b_5) h_{11}^2 + (a_8 + a_5 - b_9) h_{11}^3 + (b_6 - a_9) h_{11}^4 + a_6 h_{11}^5}{h_{11}^6} \\
d_7 = -\frac{2b_4 - a_7 h_{11}^2 + (2a_4 - b_7) h_{11}}{h_{11}^3}, d_8 = \frac{2b_4 + (2a_4 - b_7) h_{11} + (b_8 - a_7) h_{11}^2 + a_8 h_{11}^3}{h_{11}^4} \\
d_9 = -\frac{2b_4 + (2a_4 - 2b_7) h_{11} + (b_8 - 2a_7 + 2b_5) h_{11}^2 + (2a_5 + a_8 - b_9) h_{11}^3 - a_9 h_{11}^4}{h_{11}^5}.
\end{cases}
\tag{12}
$$

$$\left\{\begin{array}{c} e_0 = c_0 + b_0 h_{11}, e_1 = \frac{c_1 + b_1 h_{11}}{h_{11}}, e_2 = -\frac{c_1 + (b_1 - c_2) h_{11} - b_2 h_{11}^2}{h_{11}^2} \\ e_3 = \frac{c_1 + (b_1 - c_2) h_{11} + (c_3 - b_2) h_{11}^2 + b_3 h_{11}^3}{h_{11}^3}, e_4 = \frac{c_4 + b_4 h_{11}}{h_{11}^2} \\ e_5 = \frac{c_4 + (b_4 - c_7) h_{11} + b_5 h_{11}^3 + (c_5 - b_7) h_{11}^2}{h_{11}^4} \\ e_6 = \frac{c_4 + (b_4 - c_7) h_{11} + (c_5 - b_7 + c_8) h_{11}^2 + (b_8 + b_5 - c_9) h_{11}^3 + (c_6 - b_9) h_{11}^4 + b_6 h_{11}^5}{h_{11}^6} \\ e_7 = -\frac{2c_4 + (2b_4 - c_7) h_{11} - b_7 h_{11}^2}{h_{11}^3}, e_8 = \frac{2c_4 + (2b_4 - c_7) h_{11} + (c_8 - b_7) h_{11}^2 + b_8 h_{11}^3}{h_{11}^4} \\ e_9 = -\frac{2c_4 + (2b_4 - 2c_7) h_{11} + (c_8 - 2b_7 + 2c_5) h_{11}^2 + (b_8 - c_9 + 2b_5) h_{11}^3 - b_9 h_{11}^4}{h_{11}^5} \end{array}\right. \tag{13}$$

and

$$\left\{\begin{array}{c} f_0 = c_0 h_{11}, f_1 = c_1, f_2 = -\frac{c_1 - c_2 h_{11}}{h_{11}}, f_3 = \frac{c_1 + c_3 h_{11}^2 - c_2 h_{11}}{h_{11}^2}, f_4 = \frac{c_4}{h_{11}} \\ f_5 = \frac{c_4 - c_7 h_{11} + c_5 h_{11}^2}{h_{11}^3}, f_6 = \frac{c_4 - c_7 h_{11} + (c_5 + c_8) h_{11}^2 - c_9 h_{11}^3 + c_6 h_{11}^4}{h_{11}^5}, f_7 = -\frac{2c_4 - c_7 h_{11}}{h_{11}^2} \\ f_8 = \frac{2c_4 - c_7 h_{11} + c_8 h_{11}^2}{h_{11}^3}, f_9 = -\frac{2c_4 - 2c_7 h_{11} + (2c_5 + c_8) h_{11}^2 - c_9 h_{11}^3}{h_{11}^4}. \end{array}\right. \tag{14}$$

From the expressions of the parameters $(d_i, e_i, f_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ in (7)–(14), we conclude that there exists at least 23 forms of variations for these parameters as follows:

$$\left\{\begin{array}{c} u = \alpha, u = \alpha v + \beta \\ u = \alpha \frac{v}{w}, u = \alpha \frac{v}{w^2}, u = \frac{\alpha}{v}, u = \alpha \frac{v}{ws}, u = \frac{\alpha + \beta v}{w} \\ u = \frac{\alpha + \beta v}{w^2}, u = \frac{\alpha + \beta v}{v^2}, u = \frac{\alpha + \beta v}{ws}, u = \frac{\alpha + \beta v}{wy}, u = \frac{\alpha + \beta v}{v} \\ u = s \left(\frac{\alpha + \beta v}{v^2}\right), u = s \left(\frac{\alpha + \beta v}{v^3}\right), u = \frac{\alpha_1 + \alpha_2 v + \alpha_2 v^2}{v^3}, u = \frac{\alpha_1 + \alpha_2 v + \alpha_2 v^2}{v^2} \\ u = \frac{\alpha_1 + \alpha_2 v + \alpha_2 v^2}{w v^2}, u = w \frac{\alpha_1 + \alpha_2 v + \alpha_2 v^2}{v^4}, u = \frac{\alpha_0 + \alpha_1 v + \alpha_2 v^2 + \alpha_2 v^3}{v^k}, k = 3, 4 \\ u = \frac{\alpha_0 + \alpha_1 v + \alpha_2 v^2 + \alpha_2 v^3 + \alpha_3 v^4}{v^k}, k = 5, 6, u = \frac{\alpha_0 + \alpha_1 v + \alpha_2 v^2 + \alpha_2 v^3 + \alpha_3 v^4 + \alpha_5 v^5}{v^6} \end{array}\right. \tag{15}$$

Any variation in these curves of the parameters $(d_i, e_i, f_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ produce similar and equivalent dynamics of map (1). Thus the map (1) is structurally stable when the variation is in the forms (15).

3 Example

In this section, we give an example to validate the above analysis. Indeed, it was shown in [8] that any 3-D quadratic diffeomorphism with a quadratic inverse and constant Jacobian can be written as

$$g\left(x,y,z\right)=\begin{pmatrix} y \\ z \\ c_0+c_1x+c_2y+c_3z+c_4y^2+c_5z^2+c_6yz \end{pmatrix} \tag{16}$$

where $(c_i)_{1\leq i\leq 6}$ are the bifurcation parameters.

For the map (16), the different forms of maps equivalent to it are

$$g_1:\begin{cases} x'=y \\ y'=\frac{h_{22}z}{h_{33}} \\ z'=c_0h_{33}+\frac{c_1h_{33}x}{h_{11}}+\frac{c_2h_{33}y}{h_{22}}+c_3z+\frac{c_4h_{33}x^2}{h_{11}^2}+\frac{c_5h_{33}y^2}{h_{22}^2}+\frac{c_6z^2}{h_{33}} \end{cases} \tag{17}$$

$$g_2:\begin{cases} x'=\frac{h_{11}}{h_{22}}y \\ y'=z \\ z'=c_0h_{22}+\frac{c_1h_{22}x}{h_{11}}+c_2y+c_3z+\frac{c_4h_{22}x^2}{h_{11}^2}+\frac{c_5h_{22}y^2}{h_{22}^2}+\frac{c_6z^2}{h_{22}} \end{cases} \tag{18}$$

$$g_3:\begin{cases} x'=\frac{h_{11}}{h_{22}}y-\frac{h_{11}}{h_{22}^2}z \\ y'=c_0+\frac{c_1}{h_{11}}x+\frac{c_2y}{h_{22}}-\frac{(c_2-c_3h_{22})z}{h_{22}^2}+\frac{c_4x^2}{h_{11}^2}+\frac{c_5y^2}{h_{22}^2}-\frac{2c_5yz}{h_{22}^3} \\ z'=c_0h_{22}+\frac{c_1h_{22}x}{h_{11}}+c_2y-\frac{(c_2-c_3h_{22})z}{h_{22}}+\frac{c_4h_{22}x^2}{h_{11}^2}+\frac{c_5y^2}{h_{22}}+\frac{\left(c_5+c_6h_{22}^2\right)z^2}{h_{22}^3}-\frac{2c_5yz}{h_{22}^2} \end{cases} \tag{19}$$

$$g_4:\begin{cases} x'=y \\ y'=z \\ z'=c_0h_{11}+c_1x+c_2y+c_3z+\frac{c_4}{h_{11}}x^2+\frac{c_5}{h_{11}}y^2+\frac{c_6}{h_{11}}z^2 \end{cases} \tag{20}$$

$$g_5:\begin{cases} x'=y \\ y'=z \\ z'=c_0h_{11}+c_1x-\frac{(c_1-c_2h_{11})y}{h_{11}}+c_3z+\frac{c_4x^2}{h_{11}}+\frac{\left(c_4+c_5h_{11}^2\right)y^2}{h_{11}^3}+\frac{c_6z^2}{h_{11}}-\frac{2c_4xy}{h_{11}^2} \end{cases} \tag{21}$$

$$g:\begin{cases} x'=y \\ y'=c_0+\frac{c_1}{h_{11}}x-\frac{(c_1-c_2h_{11})y}{h_{11}^2}+e_3z+e_4x^2+e_5y^2+e_6z^2+e_7xy+e_8xz+e_9yz \\ z'=f_0+c_1x+f_2y+f_3z+f_4x^2+f_5y^2+f_6z^2+f_7xy+f_8xz+f_9yz \end{cases} \tag{22}$$

where

$$\begin{cases} e_3 = \frac{c_1 - c_2 h_{11} + c_3 h_{11}^2 + h_{11}^3}{h_{11}^3}, e_4 = \frac{c_4}{h_{11}^2}, e_5 = \frac{c_4 + c_5 h_{11}^2}{h_{11}^4} \\ e_6 = \frac{c_4 - c_7 h_{11} + c_5 h_{11}^2 + c_6 h_{11}^4}{h_{11}^6}, e_7 = -\frac{2c_4}{h_{11}^3}, e_8 = \frac{2c_4}{h_{11}^4} \\ e_9 = -\frac{2c_4 + 2c_5 h_{11}^2}{h_{11}^5}, f_0 = c_0 h_{11}, f_2 = -\frac{c_1 - c_2 h_{11}}{h_{11}} \\ f_3 = \frac{c_1 + c_3 h_{11}^2 - c_2 h_{11}}{h_{11}^2}, f_4 = \frac{c_4}{h_{11}}, f_5 = \frac{c_4 + c_5 h_{11}^2}{h_{11}^3} \\ f_6 = \frac{c_4 + c_5 h_{11}^2 + c_6 h_{11}^4}{h_{11}^5}, f_7 = -\frac{2c_4}{h_{11}^2}, f_8 = \frac{2c_4}{h_{11}^3}, f_9 = -\frac{2c_4 + 2c_5 h_{11}^2}{h_{11}^4}. \end{cases} \quad (23)$$

4 Conclusion

In this paper, the concept of structural stability was discussed for the case of 3-D quadratic mappings. We give the exact form of the small perturbation for these maps. The relevance of this result is that almost all results in the literature do not give any form for these perturbations.

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