Anti-Newtonian dynamics

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This paper describes a world in which Newton's first and second laws hold, but Newton's third law takes the form that the forces between any two objects are equal in magnitude and direction. The dynamics for such a system exhibit curious and unfamiliar features including chaos for two bodies in two spatial dimensions.
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I. INTRODUCTION

Consider a world in which Newton's first and second laws hold, but where Newton's third law takes the form that the forces between any two objects are equal in both magnitude and direction. We will refer to such a world as "anti-Newtonian," although that term has also been used to describe a general relativistic universe with a purely gravitomagnetic field.[1] Violations of Newton's third law are not uncommon and occur, for example, in the forces between moving charges when retardation effects are considered.[2,3] An anti-Newtonian force pair is unusual, but it might approximate some biological processes such as a spatial predator-prey problem in which the fox is attracted to the rabbit, but the rabbit is repelled by the fox. It is useful to study such a model not only for its pedagogical value in understanding the role of Newton's third law in classical mechanics[4] but also because there may be situations outside of physics where such a model is sensible.

II. TWO-BODY, ONE-DIMENSIONAL MOTION

For the simplest example, consider the interaction of two bodies with masses m_r and m_f in one dimension. The subscripts are chosen to represent rabbits and foxes as a mnemonic to aid in distinguishing the mass that is repelled from the one that is attracted because the symmetry to which we are accustomed is lacking. Assume that the force between them depends only on their separation $r=x_r-x_f$ according to some power γ. Such a force law allows a wide range of familiar situations including square well ($\gamma\to\infty$), springlike ($\gamma=1$), constant ($\gamma=0$), inverse square ($\gamma=-2$), and hard sphere ($\gamma\to-\infty$). From Newton's second law, we have

$$m_r\ddot{x}_r = r|r|^{\gamma-1}, \tag{1}$$

$$m_f\ddot{x}_f = r|r|^{\gamma-1}, \tag{2}$$

where without loss of generality, we have absorbed into the masses any multiplicative constant in the force by expressing the masses (assumed positive) in suitable units. These equations differ from conventional Newtonian dynamics only by the fact that the right-hand sides are equal rather than opposite.

An alternate and equivalent description is to take one of the masses as negative, in which case Newton's third law holds, but Newton's second law then implausibly predicts that the body with a negative mass accelerates in a direction opposite to the applied force. Furthermore, such a description would lead to negative kinetic energies and other difficulties that make such a description conceptually unappealing. The dynamics of objects with negative mass has been studied[5,6] usually in the context of their gravitational interaction.

If we combine Eqs. (1) and (2), the separation r obeys the differential equation

$$\ddot{r} = \left(\frac{1}{m_r} - \frac{1}{m_f}\right) r|r|^{\gamma-1}, \tag{3}$$

which can be easily integrated to obtain a relation between the relative velocity $v=\dot{x}_r-\dot{x}_f$ and the separation given for $\gamma\neq-1$ by

$$v^2 = v_0^2 + \frac{2(m_f-m_r)}{(\gamma+1)m_rm_f}(r^{\gamma+1}-r_0^{\gamma+1}), \tag{4}$$

where v_0 and r_0 (assumed positive) are the initial relative velocity and separation, respectively. The case $\gamma=-1$ is special because the second term in Eq. (4) becomes zero divided by zero. The integration for this case leads to the result

$$v^2 = v_0^2 + \frac{2(m_f-m_r)}{m_rm_f}\ln\frac{r}{r_0}. \tag{5}$$

To acclimate to the peculiar properties of this system, consider the following cases.

A. $m_r < m_f$

Consider the case in which the rabbit is less massive (and hence more mobile) than the fox. For $\gamma\geq-1$ the right-hand sides of Eqs. (4) and (5) are positive for large r and, thus, the relative velocity increases without limit as r increases, resulting from a force that does not diminish rapidly enough as the separation increases to keep the work done by the force bounded. The rabbit outruns the fox, and their separation approaches infinity with an infinite relative velocity.

For $\gamma<-1$ (as, for example, an inverse square-law force), the separation also approaches infinity but with a finite relative velocity given by

$$v_\infty = \sqrt{v_0^2 - \frac{2(m_f-m_r)}{(\gamma+1)m_rm_f}r_0^{\gamma+1}}. \tag{6}$$

In both cases the system is unbounded, and the rabbit outruns the fox.

B. $m_r=m_f$

More interesting is the case in which the rabbit and fox are equally mobile ($m_r=m_f\equiv m$), for which Eq. (4) predicts that their relative velocity is conserved ($v=v_0$), independent of γ. If they are initially at rest relative to one another ($v_0=0$) with a separation of r_0, they both obey the same equation of motion and accelerate together, like one race car pacing another, and they maintain a constant separation as their individual velocities approach infinity according to $v_r = v_f = r_0^{\gamma} t / m$.

It is evident that neither energy nor momentum is conserved in this interaction, nor does the center of mass move with constant velocity. Each mass does positive work on the other, with the rabbit pulling the fox and the fox pushing the rabbit. Such a violation of fundamental conservation laws is impossible with inert objects, but biological objects can generate the needed energy through metabolism, and the momentum comes from friction with the ground across which they are running. This example also illustrates the need for additional forces to keep the velocities bounded.

C. $m_r > m_f$

Consider the case where the rabbit is less mobile (more massive) than the fox (perhaps the prey is a lamb rather than a rabbit), in which case we expect the fox to catch the prey. From Eq. (3) the acceleration is directed opposite to the displacement, which implies that r will eventually reach zero. For $\gamma > -1$ the fox reaches the rabbit in a finite time moving with a relative velocity given by

$$v_f = \sqrt{v_0^2 + \frac{2(m_r - m_f)}{(\gamma + 1) m_r m_f} r_0^{\gamma+1}}. \tag{7}$$

If the rabbit was penetrable, the fox would overshoot and execute a periodic oscillation of constant amplitude, an example of which is the simple harmonic oscillator given by Eq. (3) with $\gamma = 1$. For $\gamma \le -1$ the relative velocity as given by Eq. (4) at $r = 0$ is infinite, as with an object falling into an infinitely dense black hole.

D. Collisions

It is clear that if the relative velocity between the rabbit and the fox is sufficiently negative (they approach one another sufficiently fast), a collision is possible. It is instructive to consider collisions in which the impulse I causes the momentum of each mass to change by the same rather than opposite amounts as required by the anti-Newtonian assumption,

$$I \equiv \int F dt = m_r \Delta v_r = m_f \Delta v_f. \tag{8}$$

In such a collision neither energy nor momentum is conserved, although there is a relation between the initial and final velocities. Without loss of generality, consider a coordinate system in which the rabbit is initially at rest at $x = 0$ and the fox is moving toward the rabbit with an initial velocity of v_0. From Eq. (8) the respective velocities after the collision are related by

$$m_f v_f - m_r v_r = m_f v_0. \tag{9}$$

The simplest situation is the perfectly inelastic collision in which $v_r = v_f \equiv v$, for which we obtain

$$v = \frac{m_f v_0}{m_f - m_r}. \tag{10}$$

The limits $m_r / m_f \to 0$ and $m_r / m_f \to \infty$ are reasonable; in the former case leading to the fox catching the rabbit and the two continuing to move with velocity v_0, and in the latter case with the fox and rabbit coming to rest. Intermediate cases are bizarre, with the velocity switching from $+\infty$ to $-\infty$ at $m_r = m_f$. What happens is that when the rabbit is infinitesimally less massive than the fox, their relative velocity just before the collision is zero, and they chase each other with ever increasing speed before eventually colliding. In contrast, if the rabbit is infinitesimally more massive than the fox, the fox slightly overshoots the rabbit and they both reverse direction and begin a prolonged chase in the backward direction, something like Wile E. Coyote trying to catch the Road Runner.

In an elastic collision the work done on the two bodies ordinarily is equal and opposite so that the total kinetic energy is conserved. The analogous condition for an anti-Newtonian elastic collision is for the work done on the two bodies to be equal but not opposite so that they both gain an equal kinetic energy. For the case where the fox is moving with an initial velocity of v_0 and the rabbit is at rest, the velocities after the collision are related by

$$W \equiv \int F dx = \frac{1}{2} m_f v_f^2 - \frac{1}{2} m_f v_0^2 = \frac{1}{2} m_r v_r^2. \tag{11}$$

If we combine this condition with Eq. (9), we obtain

$$v_f = v_0 \left(1 + \frac{2 m_r}{m_f - m_r} \right), \tag{12}$$

$$v_r = v_0 \left(2 + \frac{2 m_r}{m_f - m_r} \right). \tag{13}$$

As with the inelastic collision, the velocities of both bodies approach infinity when their masses are equal. Furthermore, the velocities approach infinity if $m_r > m_f$ or if $\gamma > -1.5$ because the net work done in those cases is infinite.

E. Dissipative case

The infinite velocities and continual input of energy suggest the need for a dissipative force to produce more realistic behavior. Consider the situation where the equations of motion are given by

$$m_r \ddot{x}_r = r |r|^{\gamma - 1} - b_r v_r, \tag{14}$$

$$m_f \ddot{x}_f = r |r|^{\gamma - 1} - b_f v_f, \tag{15}$$

in which the $-bv$ terms represent viscous drag, perhaps resulting from air resistance or friction with the vegetation through which the rabbit and fox must move.

Now the equations are of sufficient complexity that we resort to numerical solutions in anticipation of even more complicated examples to follow. In particular, we look for bounded solutions that are periodic or perhaps even chaotic.

The most complicated solution in one spatial dimension found in an extensive search is a limit cycle, one example of which is for $m_r = m_f = 1$, $b_r = 2$, $b_f = 1$, and $\gamma = 0$, as given in Fig. 1. The initial conditions are not critical but are chosen as $x_r = 0.4$, $x_f = 0.5$, $\dot{x}_r = 0$, and $\dot{x}_f = 0.3$ to be close to the attractor. Note that the fox continually overshoots the rabbit before they decelerate and resume the chase in the opposite direction.

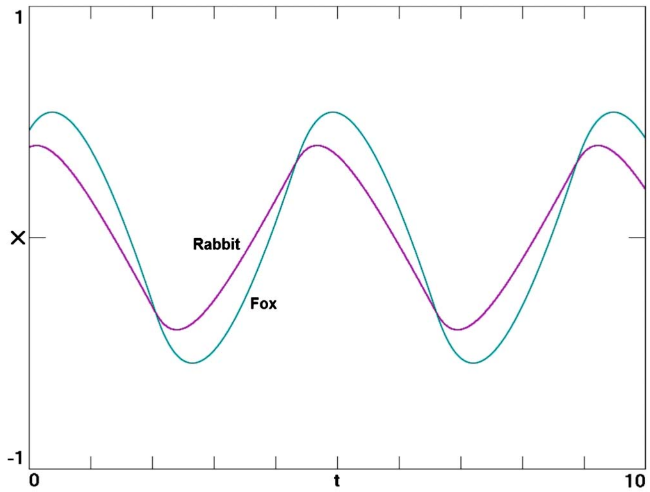

Fig. 1. Limit cycle of a rabbit and a fox in a one-dimensional chase.

III. TWO-BODY, TWO-DIMENSIONAL MOTION

If the bodies are free to move in two spatial dimensions, as the rabbit and fox would be over real horizontal terrain, more interesting dynamics are possible.

A. Conservative case

In the absence of friction, bounded dynamic solutions are possible only if the average net work done on the bodies is zero. The simplest way to arrange this case is to have the bodies orbit synchronously in concentric circles so that the force is radial and hence perpendicular to the velocity. For example, if the fox has half of the mass of the rabbit and orbits at twice the radius ($R_f = 2R_r$), then the force on them is equal and is given by $F = m_r\omega^2 R_r = m_f \omega^2 R_f$ if they have the same angular velocity ω. This result is independent of γ because their separation $r = R_f - R_r$ is constant.

There is another solution with more complicated orbits in which the positive work done on the system when the fox is in pursuit of the rabbit is just balanced by the negative work done after the fox overshoots the rabbit. One such example has $m_r = 2$, $m_f = 1$, and $\gamma = -1$, which gives the quasiperiodic trajectories shown in Fig. 2. Initial conditions are taken as $(x_r, y_r, \dot{x}_r, \dot{y}_r, x_f, y_f, \dot{x}_f, \dot{y}_f) = (1,0,0,1,2,0,0,2)$. The rabbit and fox are in synchronous, precessing orbits that periodically intersect, although there are also similar nonintersecting solutions. An animation of this case, as well as others in this paper, is available.[7]

B. Dissipative case

In the presence of friction, other interesting two-dimensional dynamics are possible. The equations that describe the motion are given by

$$\ddot{x}_r = [(x_r - x_f) r^{\gamma-1} - b_r \dot{x}_r]/m_r, \tag{16}$$

$$\ddot{y}_r = [(y_r - y_f) r^{\gamma-1} - b_r \dot{y}_r]/m_r, \tag{17}$$

$$\ddot{x}_f = [(x_r - x_f) r^{\gamma-1} - b_f \dot{x}_f]/m_f, \tag{18}$$

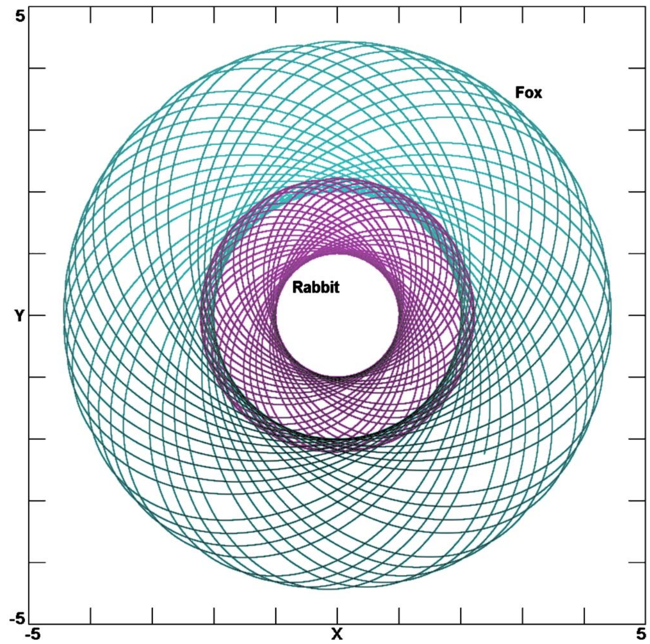

Fig. 2. Quasiperiodic trajectory of a rabbit and a fox in a two-dimensional chase with no dissipation.

$$\ddot{y}_f = [(y_r - y_f) r^{\gamma-1} - b_f \dot{y}_f]/m_f, \tag{19}$$

where $r = \sqrt{(x_r - x_f)^2 + (y_r - y_f)^2}$ is now the Euclidean distance between the bodies.

Not only are limit cycles possible but the quasiperiodic trajectories can also be attracted to a torus. One such example has $m_r = 1$, $m_f = 2$, $b_r = 1$, $b_f = 0.1$, and $\gamma = -1$, which gives the trajectories shown in Fig. 3. The initial conditions are not critical but are taken as $(x_r, y_r, \dot{x}_r, \dot{y}_r, x_f, y_f, \dot{x}_f, \dot{y}_f) = (1,1,-0.1,0,9,3,0,0.4)$ to be close to the attractor. The rabbit and fox are in synchronous, slowly precessing orbits

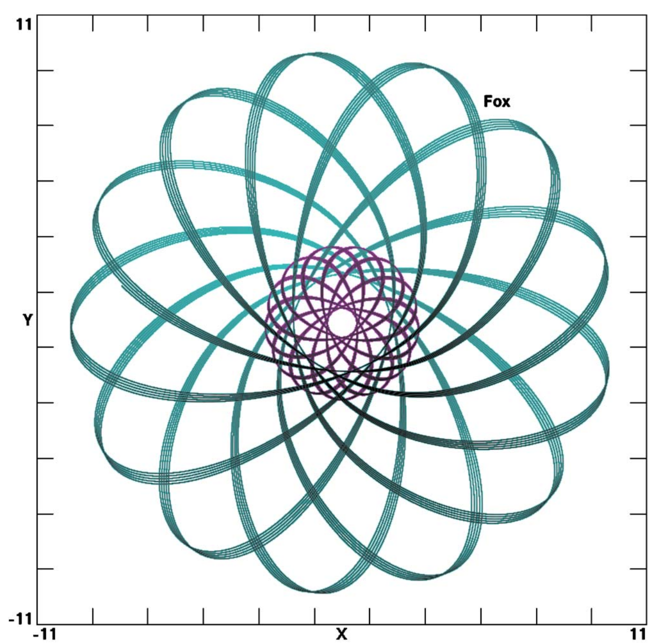

Fig. 3. Quasiperiodic trajectory of a rabbit and a fox in a two-dimensional chase with dissipation.

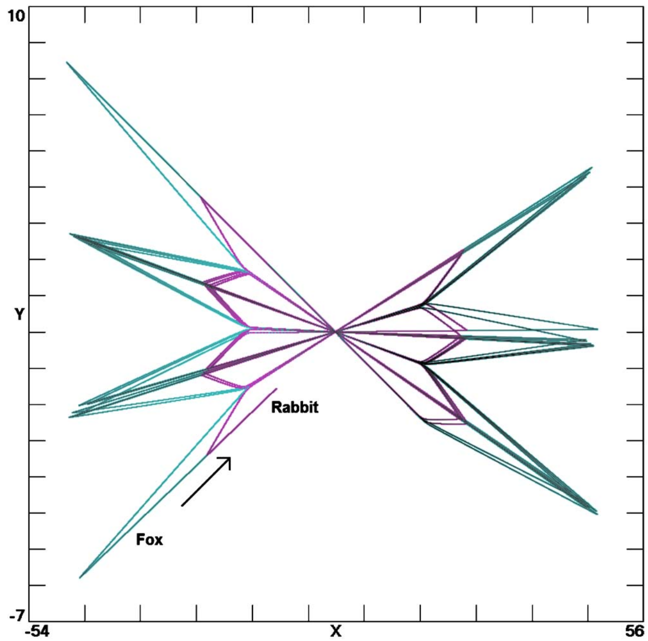

Fig. 4. Chaotic trajectory of a rabbit and a fox in a two-dimensional chase with dissipation.

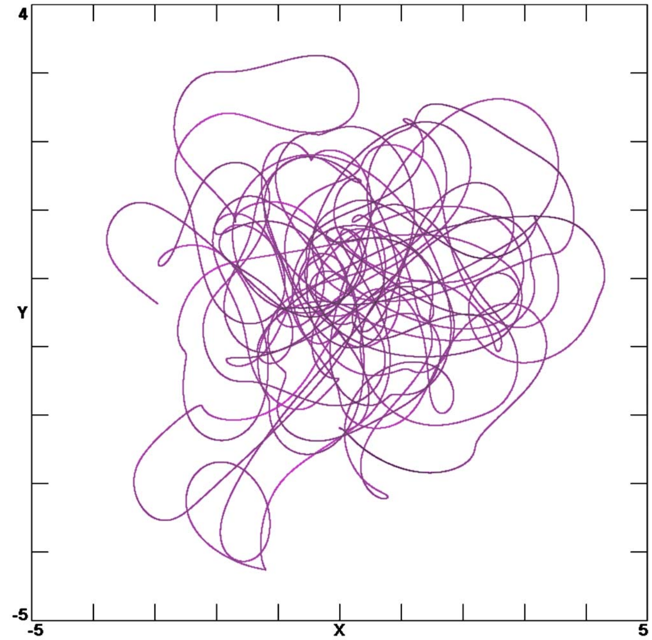

Fig. 5. Chaotic trajectory of a rabbit being chased by two identical foxes with dissipation.

with the closest approach on each pass with a value of $r=1.637\,298$. Although Figs. 2 and 3 are similar, the former is an invariant torus whose size depends on the initial conditions, while the latter is an attracting torus whose size is independent of the initial conditions (within the basin of attraction).

Chaotic solutions are also possible. One such example has $m_r=0.5$, $m_f=1$, $b_r=2$, $b_f=1$, and $\gamma=-1$, which gives the trajectories shown in Fig. 4. The initial conditions are not critical but are taken as $(x_r, y_r, \dot{x}_r, \dot{y}_r, x_f, y_f, \dot{x}_f, \dot{y}_f)=(0.3, 24, 0, 0, -0.2, 47, 0, 0)$ to be close to the strange attractor.

This unusual attractor deserves some comment. To a first approximation, the rabbit and fox oscillate back and forth along a line as in the case with one spatial dimension. The fox overtakes the rabbit on the outbound leg but misses slightly, passing either to the right or left of the rabbit by a nearly constant distance of $r\simeq 0.01$. Then they both slow nearly to a halt and resume the chase in the opposite direction, but rotated by a bit less than 6° in the plane because of their nearly equal deflection during the close encounter. The trajectory would exhibit a regular precession except for the fact that the sequence of right and left passings is apparently chaotic and has properties indistinguishable from a sequence of coin tosses, causing the orbit to walk randomly in angle, making the system ripe for analysis by symbolic dynamics.[8,9]

IV. THREE-BODY, TWO-DIMENSIONAL MOTION

The next level of complexity involves three bodies moving in two dimensions. Because of the asymmetry of the situation, there are two cases to consider—two foxes chasing one rabbit and one fox chasing two rabbits. The two bodies of the same species are assumed not to interact with one another.

A. Two foxes and one rabbit

In the frictionless case the simplest bounded dynamic solutions for two foxes chasing one rabbit have the foxes orbiting synchronously in circles concentric with the circle on which the lone rabbit orbits. The trivial solution has the foxes on top of one another and moving as a single mass as in the two-body case. The foxes can also move in concentric circles of different radii outside of and in phase with the rabbit provided that $m_rR_r=m_{f1}R_{f1}+m_{f2}R_{f2}$, independent of γ. There is another solution with the first fox in a concentric circular orbit outside the rabbit and in phase with it and the second fox in a smaller concentric circular orbit and 180° out of phase with the other two provided that $m_rR_r=m_{f1}R_{f1}-m_{f2}R_{f2}$, independent of γ. There are also solutions resembling Fig. 2 with the two foxes on top of one another.

In the presence of friction there are not only unbounded, periodic, and quasiperiodic solutions but also chaotic solutions for the case of two foxes chasing one rabbit even when the foxes have identical masses and friction. Figure 5 shows the motion of the rabbit for one such case with $m_r=1$, $m_f=2$, $b_r=3$, $b_f=1$, and $\gamma=-1$. The trajectories of the foxes are similar except that they cover more ground, but they are omitted from the figure to keep it uncluttered. The initial conditions are not critical but are taken as $(x_{r1}, y_{r1}, \dot{x}_{r1}, \dot{y}_{r1}, x_{f1}, y_{f1}, \dot{x}_{f1}, \dot{y}_{f1}, x_{f2}, y_{f2}, \dot{x}_{f2}, \dot{y}_{f2})=(0.5, 1, 0, 0, 1, 0.1, -0.4, 0.4, 0, 2, 0.4, 0)$ to be close to the strange attractor, which resides in a 12-dimensional phase space. The largest Lyapunov exponent[10] for this case is $\lambda=0.1346$. It is remarkable how different and more complicated is the trajectory from that shown in Fig. 4 when the second fox is introduced.

B. Two rabbits and one fox

For one fox chasing two rabbits without friction, the simplest bounded dynamic solutions have the fox orbiting synchronously in a circle outside of and concentric with the circles on which the rabbits orbit. The trivial solution has the rabbits on top of one another and moving as a single mass as in the two-body case. The rabbits can also move in concentric circles of different radii inside the fox's radius and in phase with it provided that the fox has a mass sufficient that $m_f R_f = m_{r1} R_{r1} + m_{r2} R_{r2}$ can be satisfied with all R's positive and R_f greater than the larger of R_{r1} and R_{r2}, independent of γ. There is no out-of-phase solution for the rabbits because all the forces must be radially inward. However, there are solutions resembling Fig. 2 with the two rabbits on top of one another.

In the presence of friction the usual situation for one fox chasing two rabbits is for at least one of the rabbits to escape to infinity while the fox is preoccupied with chasing the other, in which case the problem reduces to the two-body case. If the domain is bounded, more interesting solutions can occur, but then the size of the domain and the boundary conditions would come into play. Another possibility is to introduce a second fox, but there would still be a tendency for one of the rabbits to escape or for the motion to decouple into two spatially separated two-body problems with a different fox in pursuit of each rabbit.

V. DISCUSSION

In comparison with gravitational or Coulomb interactions, anti-Newtonian systems have much richer dynamics even with only two bodies in two spatial dimensions. In particular, they admit chaos and strange attractors because they are not constrained by energy and momentum conservation. With three bodies, chaos is even more common, suggesting that it might be the rule in the many-body limit.

Logical extensions of this work include studies of more than three bodies in three spatial dimensions, for example, to model how a swarm of prey might react to an attacking predator,[11] perhaps in a bounded domain, and to explore the regions of parameter space over which the various kinds of dynamics occur. Bodies of the same type could be taken to interact by conventional Newtonian forces,[12] perhaps with something like a van der Waals force[13] to model the tendency of the predators to hunt in packs and for the prey to flock together, thereby keeping their motion bounded.

The large velocities that often occur suggest the need for something like an "anti-Einsteinian" theory of special relativity,[14] perhaps with a much smaller limiting velocity in keeping with the biological examples to which it might apply.

Even more speculatively, the bound states, especially those in which the foxes encircle the rabbit without damping, suggest the possibility of an "anti-Bohr" atom[15] in which the forces are anti-Newtonian and the angular momentum is quantized. An electron and a proton could orbit synchronously in the presence of anti-Newtonian forces if the radius of the proton's orbit was smaller than that of the electrons by their mass ratio, which is not too different from the situation in a real atom. With the addition of something like a Pauli exclusion principle, an "anti-Periodic Table of the elements" might be concocted.

Such cases are almost completely unexplored and are of possible interest as biological models even if they are otherwise nonphysical. They have considerable pedagogical value and constitute excellent student projects.

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