Three-body Coulomb Problem Symmetric Case

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1 Equations of motion

The equations of motion for the three-body Coulomb problem can be simplified when symmetry is enforced. It is easiest to first study the problem in two spatial dimensions. We assume that the positive charges act symmetrically across the x-axis, while the negative charge is confined on the x-axis. Defining the third charge to be the negative charge, we require that $m_2 = m_1$, $q_2 = q_1$, $x_2 = x_1$, $y_2 = -y_1$, $y_3 = 0$, $\dot{x}_2 = \dot{x}_1$, $\dot{y}_2 = -\dot{y}_1$, and $\dot{y}_3 = 0$. The equations of motion then reduce to:

$$m_1\ddot{x}_1 = \frac{q_1 q_3 (x_1 - x_3)}{[(x_1 - x_3)^2 + y_1^2]^{3/2}} \tag{1}$$

$$m_1\ddot{y}_1 = \frac{q_1^2}{4y_1^2} + \frac{q_1 q_3 y_1}{[(x_1 - x_3)^2 + y_1^2]^{3/2}} \tag{2}$$

$$m_3\ddot{x}_3 = \frac{2q_1 q_3 (x_3 - x_1)}{[(x_1 - x_3)^2 + y_1^2]^{3/2}} \tag{3}$$

$$x_2 = x_1 \tag{4}$$

$$y_2 = -y_1 \tag{5}$$

$$y_3 = 0 \tag{6}$$

Furthermore, the x-position of the third charge is fixed from conservation of momentum in the center of mass frame. If $x_1 = x_3 = 0$ at some time, then:

$$m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3 = 0 \Rightarrow x_3 = -\frac{2m_1 x_1}{m_3} \tag{7}$$

Plugging this into the equations of motion, we get that:

$$m_1\ddot{x}_1 = \frac{q_1 q_3 (1 + \frac{2m_1}{m_3}) x_1}{[(1 + \frac{2m_1}{m_3})^2 x_1^2 + y_1^2]^{3/2}} \tag{8}$$

$$m_1\ddot{y}_1 = \frac{q_1^2}{4y_1^2} + \frac{q_1 q_3 y_1}{[(1 + \frac{2m_1}{m_3})^2 x_1^2 + y_1^2]^{3/2}} \tag{9}$$

$$x_2 = x_1 \tag{10}$$

$$y_2 = -y_1 \tag{11}$$

$$x_3 = -\frac{2m_1 x_1}{m_3} \tag{12}$$

$$y_3 = 0 \tag{13}$$

Hence, the dynamics are entirely contained in the motion of the first charge. Time can be scaled by $t \to \sqrt{\frac{m_1}{|q_1 q_3|}} t$ to remove a factor of m_1, q_1, and q_3. It is useful to define the following dimensionless parameters in terms of the ratios of the physical parameters:

$$M = 1 + \frac{2m_1}{m_3} \tag{14}$$

$$Q = \frac{|q_1|}{4|q_3|} \tag{15}$$

To be physical, $M > 1$ and $Q > 0$. Additionally, the initial y-position must be positive. Removing the subscripts, the effective equations of motion then take a particularly simple form:

$$\ddot{x} = \frac{-Mx}{(M^2 x^2 + y^2)^{3/2}} \tag{16}$$

$$\ddot{y} = \frac{Q}{y^2} - \frac{y}{(M^2 x^2 + y^2)^{3/2}} \tag{17}$$

2 Energy

From now on, we will use $v_x = \dot{x}$ and $v_y = \dot{y}$. The total energy can be written in terms of M and Q as:

$$E = Mv_x^2 + v_y^2 + \frac{2Q}{y} - \frac{2}{\sqrt{M^2x^2 + y^2}} \tag{18}$$

The constraint of energy conservation can be used to reduce the system by one dimension. Solving for $\dot{y}$, we have:

$$v_y = \pm\sqrt{E - Mv_x^2 - \frac{2Q}{y} + \frac{2}{\sqrt{M^2x^2 + y^2}}} \tag{19}$$

This has removed $\dot{y}$ as an independent variable. The initial condition for $\dot{y}$ has been replaced with an energy parameter E, which should be negative for bounded cases. Unfortunately, there is an ambiguity in the sign of $\dot{y}$. The sign changes as a function of time, so it is difficult to implement this equation.

3 Bounded cases

An important feature of this symmetric case is that it is bounded if $E < 0$. This can be seen as follows.

If the charge is to escape, then $M^2x^2 + y^2 \to \infty$. In this limit, the remaining terms of E are:

$$E_\infty = Mv_x^2 + v_y^2 + \frac{2Q}{y} \tag{20}$$

These terms are all positive since $y > 0$. Hence, if we start the charge with negative E and assume that it is conserved, this scenario is impossible and the particle cannot escape.

This is not true for the general three-body Coulomb problem, in which two of the charges can gain more negative energy by decreasing their separation. This can give the third charge the positive energy required to escape. In a chaotic case for the general three-body Coulomb problem, such an arrangement will eventually occur, so no chaotic system remains bounded.

4 Levi-Civita regularization

The singularity at the origin can be removed by reparameterizing the equations through Levi-Civita regularization (Diacu, 2003). This is important for avoiding numerical singularities in cases where the particles nearly collide.

The first step is to introduce variables ξ and η, defined by:

$$y = \xi^2 \tag{21}$$

$$v_y = \frac{\eta}{\xi} \tag{22}$$

Additionally, time must be reparametrized according to $\frac{dt}{ds} = 2\xi^2$. Then the energy becomes:

$$E = Mv_x^2 + \frac{\eta^2}{\xi^2} + \frac{2Q}{\xi^2} - \frac{2}{\sqrt{M^2x^2 + \xi^4}} \tag{23}$$

The equations of motion then become:

$$\dot{x} = 2\xi^2 v_x \tag{24}$$

$$\dot{\xi} = \eta \tag{25}$$

$$\dot{v}_x = \frac{-2Mx\xi^2}{(M^2x^2 + \xi^4)^{3/2}} \tag{26}$$

$$\dot{\eta} = E\xi - Mv_x^2\xi + \frac{2M^2x^2\xi}{(M^2x^2 + \xi^4)^{3/2}} \tag{27}$$

The Jacobian matrix is:

$$J = \begin{pmatrix} 0 & 4\xi v_x & 2\xi^2 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4M^3x^2\xi^2 - 2M\xi^6}{(M^2x^2+\xi^4)^{5/2}} & \frac{-4M^3x^3\xi + 8Mx\xi^5}{(M^2x^2+\xi^4)^{5/2}} & 0 & 0 \\ \frac{-2M^4x^3\xi + 4Mx\xi^5}{(M^2x^2+\xi^4)^{5/2}} & E - Mv_x^2 + \frac{2M^4x^4 - 10M^2x^2\xi^4}{(M^2x^2+\xi^4)^{5/2}} & -2Mv_x\xi & 0 \end{pmatrix}$$

At the collision point of $\xi = 0$, the Jacobian becomes:

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & E - Mv_x^2 + \frac{2}{Mx} & 0 & 0 \end{pmatrix}$$

This gives the characteristic equation:

$$\lambda^4 - \lambda^2(E - Mv_x^2 + \frac{2}{Mx}) = 0 \tag{28}$$

The quantity $E - Mv_x^2 + 2/(Mx)$ is positive, so the real eigenvalues are:

$$\lambda_1 = \lambda_2 = 0 \tag{29}$$

$$\lambda_3 = \sqrt{E - Mv_x^2 + \frac{2}{Mx}} \tag{30}$$

$$\lambda_4 = -\sqrt{E - Mv_x^2 + \frac{2}{Mx}} \tag{31}$$