Structural Stability and Hyperbolicity Violation in High-Dimensional Dynamical Systems

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This report investigates the dynamical stability conjectures of Palis and Smale, and Pugh and Shub from the standpoint of numerical observation and lays the foundation for a stability conjecture. As the dimension of a dissipative dynamical system is increased, it is observed that the number of positive Lyapunov exponents increases monotonically, the Lyapunov exponents tend towards continuous change with respect to parameter variation, the number of observable periodic windows decreases (at least below numerical precision), and a subset of parameter space exists such that topological change is very common with small parameter perturbation. However, this seemingly inevitable topological variation is never catastrophic (the dynamic type is preserved) if the dimension of the system is high enough.

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I. INTRODUCTION

Much of the work in the fields of dynamical systems and differential equations have, for the last hundred years, entailed the classification and understanding of the qualitative features of the space of differentiable mappings. A primary focus is the classification of topological differences between different systems (e.g. structural stability theory). Of course one of the primary difficulties is choosing a notion of behavior that is not so strict that it differentiates on too trivial a level, yet is strict enough that it has some meaning (Palis-Smale used topological equivalence, Pugh-Shub use ergodicity). The previous stability conjectures are with respect to any $C^r$ ($r \geq 0$ varies from conjecture to conjecture) perturbation allowing for variation of the mapping, both of the functional form (with respect to the Whitney $C^r$ topology) and of parameter variation. We will concern ourselves with the latter issue. Unlike much work involving stability conjectures, our work is numerical, and it focuses on observable asymptotic behaviors in high-dimensional systems. Our chief claim is that generally, for high-dimensional dynamical systems in our construction, there exist large portions of parameter space such that topological variation inevitably accompanies parameter variation, yet the topological variation happens in a “smooth,” non-erratic manner. Let us state our results without rigor, noting that we will save more rigorous statements for section (III).

Statement of Results 1 (Informal) Given our particular impositions (sections (II A 4) and (II A 1)) upon the space of $C^r$ discrete-time maps from compact sets to themselves, and an invariant measure (used for calculating Lyapunov exponents), in the limit of high dimension, there exists a subset of parameter space such that strict hyperbolicity is violated on a nearly dense (and hence unavoidable), yet zero-measure (with respect to Lebesgue measure), subset of parameter space.

A more refined version of this statement will contain all of our results. For mathematicians, we note that although the stability conjecture of Palis and Smale [1] is quite true (as proved by Robbin [2], Robinson [3], and Mañé [4]), we show that in high dimensions, this structural stability may occur over such small sets in the parameter space that it may never be observed in chaotic regimes of parameter space. Nevertheless, this lack of observable structural stability has very mild consequences for applied scientists.

A. Outline

As this paper is attempting to reach a diverse readership, we will briefly outline the work for ease of reading. Of the remaining introduction sections, section (IB) can be skipped by readers familiar with the stability conjecture of Smale and Palis and the stable ergodicity of Pugh and Shub.

Following the introduction we will address various preliminary topics pertaining to this report. Beginning in section (II A 1), we present the mathematical justification for the study of time-delay maps being sufficient for a general study of $d > 1$ dimensional dynamical systems. This section is followed with a discussion of neural networks, beginning with their definition in the abstract (section (II A 2)). Following the definition of neural networks, we explain the mappings neural networks are able

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to approximate (section (II A 3)). In section (II A 4) we give our specific construction of neural networks. Those uninterested in the mathematical justifications for our models and only interested in our specific formulation should skip sections (II A 1) thru (II A 3) and concentrate on section (II A 4). The discussion of our set of mappings is followed by relevant definitions from hyperbolicity and ergodic theory (section (II B)). It is here where we define the Lyapunov spectrum, hyperbolic maps, and discuss relevant stability conjectures. Section (II C) provides justification for our use of Lyapunov exponent calculations upon our space of mappings (the neural networks). Readers familiar with topics in hyperbolicity and ergodic theory can skip this section and refer to it as is needed for an understanding of the results. Lastly, in section (II D), we make a series of definitions we will need for our numerical arguments. Without an understanding of these definitions, it is difficult to understand both our conjectures of our arguments.

Section (III) discusses the conjectures we wish to investigate formally. For those interested in just the results of this report, reading sections (II D), (III) and (VII) will suffice. The next section, section (IV), discusses the errors present in our chief numerical tool, the Lyapunov spectrum. This section is necessary for a fine and careful understanding of this report, but this section is easily skipped upon first reading. We then begin our preliminary numerical arguments. Section (V), addresses the three major properties we need to argue for our conjectures. For an understanding of our arguments and why our conclusions make sense, reading this section is necessary. The main arguments regarding our conjectures follow in section (VI). It is in this section that we make the case for the claims of section (III). The summary section (section (VII)) begins with a summary of our numerical arguments and how they apply to our conjectures. We then interpret our results in light of various stability conjectures and other results from the dynamics community.

B. Background

To present a full background with respect to the topics and motivations for our study would be out of place in this report. We will instead discuss the roots of our problems and a few relevant highlights, leaving the reader with references to the survey papers of Burns et. al [5], Pugh and Shub [6], Palis [7], and Nitecki [8] for a more thorough introduction.

The origin of our work, as with all of dynamical systems, lies with Poincaré who split the study of dynamics in mathematics into two categories, conservative and dissipative systems; we will be concerned with the latter. We will refrain from beginning with Poincaré and instead begin in the 1960’s with the pursuit of the “lost dream.”

The “dream” amounted to the conjecture that structurally stable dynamical systems would be dense among all dynamical systems. For mathematicians, the dream was motivated primarily by a desire to classify dynamical systems via their topological behavior. For physicists and other scientists however, this dream was two-fold. First, since dynamical systems (via differential equations and discrete-time maps) are usually used to model physical phenomena, a geometrical understanding of how these systems behave in general is, from an intuitive standpoint, very insightful. However, there is a more practical motivation for the stability dream. Most experimental scientists who work on highly nonlinear systems (e.g. plasma physics and fluid dynamics) are painfully aware of the existence of the dynamic stability that the mathematicians where hoping to capture with the stability conjecture of Palis and Smale. When we write dynamic stability we do not mean fixed point versus chaotic dynamics, rather we mean that upon normal or induced experimental perturbations, dynamic types are highly persistent. Experimentalists have been attempting to control and eliminate turbulence and chaos since they began performing experiments — it is clear from our experience that turbulence and chaos are highly stable with respect to perturbations in highly complicated dynamical systems, the why and how of the stability and what is the right notion of equivalence to capture that stability is the question. In a practical sense, the hope lies in that, if the geometric characteristics that allow chaos to persist can be understood, it might be easier to control or even eliminate those characteristics. At the very least, it would be useful to at least know very precisely why we can’t control or rid our systems of turbulent behavior. At any rate, the dream was “lost” in the late 1960’s via many counter examples ([1]), leaving room for a very rich theory. Conjectures regarding weaker forms of the dream for which a subset of “nice” diffeomorphisms would be dense were put forth, many lasted less than a day, and none worked. The revival of the dream in the 1990’s involved a different notion of nice - stable ergodicity.

Near the time of the demise of the "dream" the notion of structural stability together with Smale’s notion of hyperbolicity was used to formulate the stability conjecture (the connection between structural stability and hyperbolicity - now a theorem) [9]. The stability conjecture says that “a system is $C^r$ stable if its limit set is hyperbolic and, moreover, stable and unstable manifolds meet transversally at all points.” [7]

To attack the stability conjecture, Smale had introduced axiom A. Dynamical systems that satisfy axiom A are strictly hyperbolic (definition (6)) and have dense periodic points on the non-wandering set[63]. A further condition that was needed is the strong transversality condition - $f$ satisfies the strong transversality condition when, on every $x \in M$, the stable and unstable manifolds $W^s_x$ and $W^u_x$ are transverse at $x$. That axiom A and strong transversality imply $C^r$ structural stability was shown by Robbin [2] for $r \geq 2$ and Robinson [3] for $r = 1$. The other direction of the stability conjecture was much more elusive, yet in 1980 this was shown by Mañé [4] for $r = 1$. 


Nevertheless, due to many examples of structurally unstable systems being dense amongst many “common” types of dynamical systems, proposing some global structure for a space of dynamical systems became much more unlikely. Newhouse [10] was able to show that infinitely many sinks occur for a residual subset of an open set of $C^2$ diffeomorphisms near a system exhibiting a homoclinic tangency. Further, it was discovered that orbits can be highly sensitive to initial conditions [11], [12], [13], [14]. Much of the sensitivity to initial conditions was investigated numerically by non-mathematicians. Together, the examples from both pure mathematics and the sciences sealed the demise of the “dream” (via topological notions), yet they opened the door for a wonderful and diverse theory. Nevertheless, despite the fact that structural stability does not capture all we wish it to capture, it is still a very useful, intuitive tool.

Again, from a physical perspective, the question of the existence of dynamic stability is not open - physicists and engineers have been trying to suppress chaos and turbulence in high-dimensional systems for several hundred years. The trick in mathematics is writing down a relevant notion of dynamic stability and then the relevant necessary geometrical characteristics to guarantee dynamic stability. From the perspective of modeling nature, structural stability says that if one selects (fits) a model equation, small errors will be irrelevant since small $C^r$ perturbations will yield topologically equivalent models. It is the topological equivalence that is too strong a characteristic for structural stability to apply to the broad range of systems we wish it to apply to. Structural stability is difficult to use in a very practical way because it is very difficult to show (or disprove the existence of) topological ($C^0$) equivalence of a neighborhood of maps. Hyperbolicity can be much easier to handle numerically, yet it is not always common. Luckily, to quote Pugh and Shub [15], “a little hyperbolicity goes a long way in guaranteeing stably ergodic behavior.” This thesis has driven the partial hyperbolicity branch of dynamical systems and is our claim as well. We will define precisely what we mean by partial hyperbolicity and will discuss relevant results a la stable ergodicity and partial hyperbolicity.

Our investigation will, in a practical, computational context, investigate the extent to which ergodic behavior and topological variation (versus parameter variation) behave given a “little bit” of hyperbolicity. Further, we will investigate one of the overall haunting questions: how much of the space of bounded $C^r$ ($r > 0$) systems is hyperbolic, and how many of the pathologies found by Newhouse and others are observable (or even existent) in the space of bounded $C^r$ dynamical systems. Stated more generally, how does hyperbolicity (and thus structural stability) “behave” in a space of bounded $C^r$ dynamical systems.

II. DEFINITIONS AND PRELIMINARIES

In this section we will define the following items: the family of dynamical systems we wish to investigate; the function space we will use in our experiments; Lyapunov exponents; and finally we will list definitions specific to our numerical arguments. The choice of scalar neural networks as our maps of choice is motivated by their being “universal approximators.”

A. Our space of mappings

The motivation and construction of the set of mappings we will use for our investigation of dynamical systems follows via two directions, the embedding theorem of Takens ([16], [17]) and the neural network approximation theorems of Hornik, Stinchomebe, and White [18]. We will use the Takens embedding theorem to demonstrate how studying time-delayed maps of the form $f : R^d \rightarrow R$ is a natural choice for studying standard dynamical systems of the form $F : R^d \rightarrow R^d$. This is important as we will be using time-delayed scalar neural networks for our study. The neural network approximation theorems show that neural networks of a particular form are open and dense in several very general sets of functions and thus can be used to approximate any function in the allowed function spaces.

There is overlap, in a sense, between these two constructions. The embedding theory shows an equivalence or the approximation capabilities of scalar time-delay dynamics with standard, $x_{t+1} = F(x_t)$ ($x_t \in R^d$) dynamics. There is no mention of, in a practical sense, the explicit functions in the Takens construction. The neural network approximation results show in a precise and practical way, what a neural network is, and what functions it can approximate. It says that neural networks can approximate the $C^r$ ($R^d$) mappings and their derivatives, but there is no mention of the time-delays we wish to use. Thus we need to discuss both the embedding theory and the neural network approximation theorems.

Those not interested in the mathematical justification of our construction may skip to section (II.A.4) where we define, in a concrete manner, our neural networks.

1. Dynamical systems construction

We wish, in this report, to investigate dynamical systems on compact sets. Specifically, begin with a compact manifold $M$ of dimension $d$ and a diffeomorphism $F \in C^r(M)$ for $r \geq 2$ defined as:

$$x_{t+1} = F(x_t)$$

with $x_t \in M$. However, for computational reasons, we will be investigating this space with neural networks that
can approximate (see section (II A 3)) dynamical systems $f \in C^r(R^d, R)$ that are time-delay maps given by:

$$y_{t+1} = f(y_t, y_{t-1}, \ldots, y_{t-(d-1)})$$  \hspace{1cm} (2)

where $y_t \in R$. Both systems (1) and (2) form dynamical systems. However, since we intend to use systems of the form (2) to investigate the space of dynamical systems as given in equation (1), we must show that a study of mappings of the form (2) is somehow equivalent to mappings of the form (1). We will demonstrate this by employing an embedding theorem of Takens to demonstrate the relationship between time-delay maps and non-time-delay maps in a more general and formal setting.

We call $g \in \hat{C}^k(M, R^n)$ an embedding if $k \geq 1$ and if the local derivative map (the Jacobian - the first order term in the Taylor expansion) is one-to-one for every point $x \in M$ (i.e. $g$ must be an immersion). The idea of the Takens embedding theorem is that given a $d$-dimensional dynamical system and a “measurement function,” $E : M \rightarrow R$ ($E$ is a $C^k$ map), where $E$ represents some empirical style measurement of $F$, there is a Takens map (which does the embedding) $g$ for which $x \in M$ can be represented as a $2d + 1$ tuple $(E(x), E \circ F(x), E \circ F^2(x), \ldots, E \circ F^{2d}(x))$ where $F$ is an ordinary difference equation (time evolution operator) on $M$. Note that the $2d + 1$ tuple is a time-delay map of $x$. We can now state the Takens embedding theorem:

**Theorem 1 (Takens’ embedding theorem [16] [17])** Let $M$ be a compact manifold with dimension $d$. There is an open dense subset $S \subset \text{Diff}(M) \times \hat{C}^k(M, R)$ with the property that the Takens map

$$g : M \rightarrow R^{2d+1}$$  \hspace{1cm} (3)

given by $g(x) = (E(x), E \circ F(x), E \circ F^2(x), \ldots, E \circ F^{2d}(x))$ is an embedding of $C^k$ manifolds, when $(F, E) \in S$.

Here $\text{Diff}(M)$ is the space of $C^k$ diffeomorphisms from $M$ to itself with the subspace topology from $C^k(M, M)$. Thus, there is an equivalence between time-delayed Takens maps of “measurements” and the “actual” dynamical system operating in time on $x_t \in M$. This equivalence is that of an embedding (the Takens map), $g : M \rightarrow R^{2d+1}$.

To demonstrate how this applies to our circumstances, consider figure (1) in which $F$ and $E$ are as given above and the embedding $g$ is explicitly given by:

$$g(x_t) = (E(x_t), E(F(x_t)), \ldots, E(F^{2d}(x_t)))$$  \hspace{1cm} (4)

In a colloquial, experimental sense, $\tilde{F}$ just keeps track of the observations from the measurement function $E$, and, at each time step, shifts the newest observation into the $2d+1$ tuple and sequentially shifts the scalar observation at time $t$ ($y_t$) of the $2d+1$ tuple to the $t-1$ position of the $2d+1$ tuple. In more explicit notation, $\tilde{F}$ is the following mapping:

$$(y_1, \ldots, y_{2d+1}) \mapsto (y_2, \ldots, y_{2d+1}, g(F(g^{-1}(y_1, \ldots, y_{2d+1}))))$$  \hspace{1cm} (5)

where, again, $\tilde{F} = g \circ F \circ g^{-1}$. The neural networks we will propose in the sections that follow can approximate $\tilde{F}$ and its derivatives (to any order) to arbitrary accuracy (a notion we will make precise later).

Let us summarize what we are attempting to do: we wish to investigate dynamical systems given by (1) but for computational reasons we wish to use dynamical systems given by (2); the Takens embedding theorem says that dynamical systems of the form (1) can be generically represented (via the Takens embedding map $g$) by time-delay dynamical systems of the form (5). Since neural networks will approximate dynamical systems of the form (5) on a compact and metrizable set, it will suffice for our investigation of dynamical systems of the form (1) to consider the space of neural networks mapping compact sets to compact sets as is given in section (II A 2).

2. Abstract neural networks

Begin by noting that, in general, a neural network is a $C^r$ mapping $\gamma : R^n \rightarrow R$. More specifically, the set of feedforward networks with a single hidden layer, $\Sigma(G)$, can be written:

$$\Sigma(G) \equiv \{ \gamma : R^d \rightarrow R | \gamma(x) = \sum_{i=1}^{N} \beta_i G(\tilde{x}^T \omega_i) \}$$  \hspace{1cm} (6)

where $x \in R^d$, is the $d-$vector of networks inputs, $\tilde{x}^T \equiv (1, x^T)$ (where $x^T$ is the transpose of $x$), $N$ is the number of hidden units (neurons), $\beta_1, \ldots, \beta_N \in R$ are the hidden-to-output layer weights, $\omega_1, \ldots, \omega_N \in R^{d+1}$ are the input-to-hidden layer weights, and $G : R^d \rightarrow R$ is the hidden layer activation function (or neuron). The partial derivatives of the network output function, $\gamma$, are

$$\frac{\partial g(x)}{\partial x_k} = \sum_{i=1}^{N} \beta_i \omega_{ik} DG(\tilde{x}^T \omega_i)$$  \hspace{1cm} (7)

where $x_k$ is the $k^{th}$ component of the $x$ vector, $\omega_{ik}$ is the $k^{th}$ component of $\omega_i$, and $DG$ is the usual first derivative of $G$. The matrix of partial derivatives (the Jacobian) takes a particularly simple form when the $x$ vector is a sequence of time delays $(x_t = (y_t, y_{t-1}, \ldots, y_{t-(d-1)})$ for $x_t \in R^d$ and $y_t \in R$). It is for precisely this reason that we choose the time-delayed formulation.

3. Neural networks as function approximations

We will begin with a brief description of spaces of maps useful for our purposes and conclude with the keynote theorems of Hornik et al. [18] necessary for our work. Hornik et al. provided the theoretical justification for the use of neural networks as function approximators. The aforementioned authors provide a degree of generality that we will not need; for their results in full generality see [19], [18].
The ability of neural networks to approximate functions which are of particular interest, can be most easily seen via a brief discussion of Sobolev function space, $S^m_p$. We will be brief, noting references Adams [20] and Hebey [21] for readers wanting more depth with respect to Sobolev spaces. For the sake of clarity and simplification, let us make a few remarks which will pertain to the rest of this section:

i. $\mu$ is a measure; $\lambda$ is the standard Lebesgue measure; for all practical purposes, $\mu = \lambda$;

ii. $l$, $m$ and $d$ are finite, non-negative integers; $m$ will be with reference to a degree of continuity of some functions spaces and $d$ will be the dimension of the space we are operating upon;

iii. $p \in \mathbb{R}$, $1 \leq p < \infty$; $p$ will be with reference to a norm — either the $L_p$ norm or the Sobolev norm;

iv. $U \subset \mathbb{R}^d$, $U$ is measurable.

v. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)^T$ is a $d$-tuple of non-negative integers (or a multi-index) satisfying $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $|\alpha| \leq m$;

vi. for $x \in \mathbb{R}^d$, $x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_d^{\alpha_d}$.

vii. $D^\alpha$ denotes the partial derivative of order $|\alpha|$

\[
\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{(\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d})}
\]  

viii. $u \in L^1_{loc}(U)$ be a locally integrable, real valued function on $U$

ix. $\rho_{p,m}^n$ is a metric, dependent upon the subset $U$, the measure $\mu$, and $p$ and $m$ in a manner we will define shortly;

x. $\| \cdot \|$ is the standard norm in $L_p(U)$.

Let $m$ be a positive integer and $1 \leq p < \infty$, we define the Sobolev norm, $\| \cdot \|_{m,p}$, as follows:

\[
\|u\|_{m,p} = \left( \sum_{0 \leq |\alpha| \leq m} (\| D^\alpha u \|_p^p) \right)^{1/p}
\]  

for $u \in L^1_{loc}(U)$ be a locally integrable, real valued function on $U \subset \mathbb{R}^d$ ($u$ could be significantly more general) and $\| \cdot \|$ is the standard norm in $L_p(U)$. Likewise, the Sobolev metric can be defined:

\[
\rho_{p,m}^n(f,g) \equiv \|f-g\|_{m,p,U,\mu}
\]  

It is important to note that this metric is dependent upon $U$.

For ease of notation, let us define the set of $m$ times differentiable functions on $U$,

\[
C^m(U) = \{ f \in C(U) | D^\alpha f \in C(U), \| D^\alpha f \|_p < \infty \forall \alpha, |\alpha| \leq m \}
\]  

We are now free to define the Sobolev space for which our results will apply.

**Definition 1** For any positive integer $m$ and $1 \leq p < \infty$, we define a Sobolev space $S^m_p(U, \lambda)$ as the vector space on which $\| \cdot \|_{m,p}$ is a norm:

\[
S^m_p(U, \lambda) = \{ f \in C^m(U) | \| D^\alpha f \|_{p,U,\lambda} < \infty \text{ for all } |\alpha| \leq m \}
\]
Equipped with the Sobolev norm, $S^m_p$ is a Sobolev space over $U \subset \mathbb{R}^d$.

Two functions in $S^m_p(U, \lambda)$ are close in the Sobolev metric if all the derivatives of order $0 \leq |\alpha| < m$ are close in the $L_p$ metric. It is useful to recall that we are attempting to approximate $\tilde{F} = g \circ F \circ g^{-1}$ where $\tilde{F} : \mathbb{R}^{2d+1} \rightarrow \mathbb{R}$; for this task the functions from $S^m_p(U, \lambda)$ will serve us quite nicely. The whole point of all of this machinery is to state approximation theorems that require specific notions of density, otherwise we would refrain and instead use the standard notion of $C^k$ functions — the functions that are $k$ times differentiable uninhibited with a notion of a metric or norm.

Armed with a specific function space for which the approximation results apply (there are many more), we will conclude this section by briefly stating one of the approximation results. However, before stating the approximation theorem we need two definitions — one which makes the notion of closeness of derivatives more precise and one which gives the sufficient conditions for the activation functions to perform the approximations.

**Definition 2 ($m$-uniformly dense)** Assume $m$ and $l$ are non-negative integers $0 \leq m \leq l$, $U \subset \mathbb{R}^d$, and $S \subset C^l(U)$. If, for any $f \in S$, compact $K \subset U$, and $\epsilon > 0$ there exists a $g \in \Sigma(G)$ such that:

$$\max_{|\alpha| \leq m} \sup_{x \in K} |D^\alpha f(x) - D^\alpha g(x)| < \epsilon$$

then $\Sigma(G)$ is $m$-uniformly dense on compacta in $S$.

It is this notion of $m$-uniformly dense in $S$ that provides all the approximation power of both the mappings and the derivatives (up to order $l$) of the mappings. Next we will supply the condition upon our activation function necessary for the approximation results.

**Definition 3 ($l$-finite)** Let $l$ be an non-negative integer. $G$ is said to be $l$-finite $G \in C^l(R)$ and if:

$$0 < \int |D^l G| d\lambda < \infty$$

i.e. the $l$th derivative of $G$ must be both bounded away from zero, and finite for all $l$ (recall $d\lambda$ is the standard Lebesgue volume element).

The hyperbolic tangent, our activation function, is $l$-finite.

With these two notions, we can state one of the many existing approximation results.

**Corollary 1 (corollary 3.5 [18])** If $G$ is $l$-finite, $0 \leq m \leq l$, and $U$ is an open subset of $\mathbb{R}^d$, then $\Sigma(G)$ is $m$-unformly dense on compacta in $S^m_p(U, \lambda)$ for $1 \leq p < \infty$.

In general, we wish to investigate differentiable mappings of compact sets to themselves. Further, we wish for the derivatives to be finite almost everywhere, thus the space $S^m_p(U, \lambda)$ will suffice for our purposes. Our results also apply to piecewise differentiable mappings, however this requires a more general Sobolev space, $W^m_p$. We have refrained delving into the definition of this space as it requires a bit more formalism, for those interested see [18] and [20].

### 4. Our neural network construction

The single layer feed-forward neural networks ($\gamma$’s from the above section) we will consider are of the form

$$x_t = \beta_0 + \sum_{i=1}^{N} \beta_i G \left( s \omega_{i0} + \sum_{j=1}^{d} \omega_{ij} x_{t-j} \right)$$

which is a map from $\mathbb{R}^d$ to $\mathbb{R}$. The squashing function $G$, for our purpose will be tanh(). In (15), $N$ represents the number of hidden units or neurons, $d$ is the input or embedding dimension of the system which functions simply as the number of time lags, and $s$ is a scaling factor on the weights.

The parameters are real $(\beta_i, w_{ij}, x_j, s \in \mathbb{R})$ and the $\beta_i$’s and $w_{ij}$’s are elements of weight matrices (which we hold fixed for each case). The initial conditions are denoted as $(x_0, x_1, \ldots, x_d)$, and $(x_t, x_{t+1}, \ldots, x_{t+d})$ represent the current state of the system at time $t$.

We assume that the $\beta_i$’s are $iid$ uniform over $[0, 1]$ and then re-scaled to satisfy the following condition $\sum_{i=1}^{N} \beta_i^2 = N$ The $w_{ij}$’s are iid normal with zero mean and unit variance. The $s$ parameter is a real number and can be interpreted as the standard deviation of the $w$ matrix of weights. The initial $x_i$’s are chosen iid uniform on the interval $[-1, 1]$. All the weights and initial conditions are selected randomly using a pseudo-random number generator [22], [23].

We would like to make a few notes with respect to our squashing function, tanh(). First, tanh($x$), for $|x| \gg 1$ will tend to behave much like a binary function. Thus, the states of the neural network will tend toward the finite set $(\beta_0 \pm \beta_1 \pm \beta_2 \cdots \pm \beta_n)$, or a set of $2^n$ different states. In the limit where the arguments of tanh() become infinite, the neural network will have periodic dynamics. Thus, if $\beta > s$ become very large, the system will have a greatly reduced dynamic variability. Based on this problem, one might feel tempted to bound the $\beta$’s a la $\sum_{i=1}^{N} |\beta_i| = k$ fixing $k$ for all $N$ and $d$. This is a bad idea however since, if the $\beta$’s are restricted to a sphere of radius $k$, as $N$ is increased, $(\beta_i^2)$ goes to zero [24]. The other extreme of our squashing also yields a very specific behavior type. For $x$ very near zero, the tanh($x$) function is nearly linear. Thus choosing $s$ small will force the dynamics to be mostly linear, again yielding fixed point and periodic behavior (no chaos). Thus the scaling parameter $s$ will provide a unique bifurcation parameter that will sweep from linear ranges to highly non-linear ranges, to binary ranges - fixed points to chaos and back to periodic phenomena.
Note that in a very practical sense, the measure we are imposing on the set of neural networks is our means of selecting the weights that define the networks. This will introduce a bias into our results that is unavoidable in such experiments; the very act of picking networks out of the space will determine, to some extent, our results. Unlike actual physical experiments, we could, in principle, prove an invariance of our results to our induced measure. This is difficult and beyond the scope of this paper. Instead it will suffice for our purposes to note specifically what our measure is (our weight selection method), and how it might bias our results. Our selection method will include all possible networks, but clearly not with the same likelihood. In the absence of a theorem with respect to an invariance of our induced measure, we must be careful in stating what our results imply about the ambient function space.

### B. Characteristic Lyapunov exponents and Hyperbolicity

Let us now define the diagnostics for our numerical simulations. We will begin by defining structural stability and its relevant notion of topological equivalence (between orbits, attractors, etc), topological conjugacy. We will then discuss notions that are more amenable to a numerical study, yet can be related to the geometrical notions of structural stability. Hyperbolicity will be defined in three successive definitions, each with increasing generality, culminating with a definition of partial hyperbolicity. This will be followed with a global generalization of local eigenvalues, the Lyapunov spectrum. We will include here a brief statement regarding the connection between structural stability, hyperbolicity, and the Lyapunov spectrum.

**Definition 4 (Structural Stability)** A $C^r$ discrete-time map, $f$, is structurally stable if there is a $C^r$ neighborhood, $V$ of $f$, such that any $g \in V$ is topologically conjugate to $f$, i.e., for every $g \in V$, there exists a homeomorphism $h$ such that $f = h^{-1} \circ g \circ h$.

In other words, a map is structurally stable if, for all other maps $g$ in a $C^r$ neighborhood, there exists a homeomorphism that will map the domain of $f$ to the domain of $g$, the range of $f$ to the range of $g$, and the inverses respectively. This is a purely topological notion.

Next, let us begin by defining hyperbolicity in an intuitive manner, followed by a more general definition useful for our purposes. Let us start with a linear case:

**Definition 5 (Hyperbolic linear map)** A linear map of $\mathbb{R}^n$ is called hyperbolic if all of its eigenvalues have modulus different from one.

The above definition can be generalized as follows:

**Definition 6 (Hyperbolic map)** A discrete-time map $f$ is said to be hyperbolic on a compact invariant set $\Lambda$ if there exists a continuous splitting of the tangent bundle, $TM|_\Lambda = E^s \oplus E^u$, and there are constants $C > 0$, $0 < \lambda < 1$, such that $||Df^n|_{E^s}|| < C\lambda^n$ and $||Df^{-n}|_{E^u}|| < C\lambda^n$ for any $n > 0$ and $x \in \Lambda$.

Here the stable bundle $E^s$ (respectively unstable bundle $E^u$) of $x \in \Lambda$ is the set of points $p \in M$ such that $|f^k(x) - f^k(p)| \to 0$ as $k \to -\infty$ ($k \to \infty$ respectively).

As previously mentioned, strict hyperbolicity is a bit restrictive; thus let us make precise the notion of a “little bit” of hyperbolicity:

**Definition 7 (Partial hyperbolicity)** The diffeomorphism $f$ of a smooth Riemannian manifold $M$ is said to be partially hyperbolic if for all $x \in M$ the tangent bundle $T_x M$ has the invariant splitting:

$$T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

into strong stable $E^s(x) = E^s_f(x)$, strong unstable $E^u(x) = E^u_f(x)$, and central $E^c(x) = E^c_f(x)$ bundles, at least two of which are non-trivial[64]. Thus there will exist numbers $0 < a < b < 1 < c < d$ such that, for all $x \in M$:

$$v \in E^s(x) \Rightarrow d|v| \leq ||D_x f(v)||$$

$$v \in E^c(x) \Rightarrow b|v| \leq ||D_x f(v)|| \leq c|v|$$

$$v \in E^u(x) \Rightarrow ||D_x f(v)|| \leq a|v|$$

More specific characteristics and definitions can be found in references [25], [26], [15], [5], and [27]. The key provided by definition 7 is the allowance of center bundles, zero Lyapunov exponents, and in general, neutral directions, which are not allowed in strict hyperbolicity. Thus we are allowed to keep the general framework of good topological structure, but lose structural stability. With non-trivial partial hyperbolicity (i.e. $E^c$ is not null), stable ergodicity replaces structural stability as the notion of dynamic stability in the Pugh-Shub stability conjecture (conjecture (6) of [28]). Thus what is left is to again attempt to show the extent to which stable ergodicity persists, and topological variation is not pathological, under parameter variation with non-trivial center bundles present. Again, we note that results in this area will be discussed in a later section.

In numerical simulations we will never observe an orbit on the unstable, stable, or center manifolds. Thus we will need a global notion of stability averaged along a given orbit (which will exist under weak ergodic assumptions). The notion we seek is captured by the spectrum of Lyapunov exponents.

We will initially define Lyapunov exponents formally, followed by a more practical, computational definition.

**Definition 8 (Lyapunov Exponents)** Let $f : M \to M$ be a diffeomorphism (i.e. discrete time map) on a compact Riemannian manifold of dimension $m$. Let $|| \cdot ||$ be the norm on the tangent vectors induced by the Riemannian metric on $M$. For every $x \in M$ and $v \in T_x M$
Lyapunov exponent at $x$ is denoted:

$$\chi(x,v) = \limsup_{t \to \infty} \frac{1}{t} \log||Df^n v||$$

(20)

Assume the function $\chi(x,\cdot)$ has only finitely many values on $T_x M \setminus \{0\}$ (this assumption may not be true for our dynamical systems) which we denote $\chi(x) = \chi_1(x) \leq \chi_2(x) \leq \cdots \leq \chi_m(x)$. Next denote the filtration of $T_x M$ associated with $\chi(x,\cdot)$ by $\{0\} = V_0(x) \subseteq V_1(x) \subseteq \cdots \subseteq V_m(x) = T_x M$, where $V_i(x) = \{v \in T_x M | \chi(x,v) \leq \chi_i(x)\}$. The number $k_i = \dim(V_i(x)) - \dim(V_{i-1}(x))$ is the multiplicity of the exponent $\chi_i(x)$. In general, for our networks over the parameter range we are considering, $k_i = 1$ for all $0 < i < m$. Given the above, the Lyapunov spectrum for $f$ at $x$ is defined:

$$\text{Sp}(x) = \{\chi_j^k(x)|1 \leq i \leq m\}$$

(21)

(For more information regarding Lyapunov exponents and spectra see [29], [30], or [31].)

A more computationally motivated formula for the Lyapunov exponents is given as:

$$\chi_j = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \ln \langle \langle (Df_k \delta x_j)^T, (Df_k \delta x_j) \rangle \rangle$$

(22)

where $\langle \cdot \rangle$ is the standard inner product, $\delta x_j$ is the $j$th component of the $x$ variation[65] and $Df_k$ is the “orthogonalized” Jacobian of $f$ at the $k$th iterate of $f(x)$. Through the course of our discussions we will dissect equation (22) further. It should also be noted that Lyapunov exponents have been shown to be independent of coordinate system, thus the specifics of our above definition do not affect the outcome of the exponents.

The existence of Lyapunov exponents is established by a multiplicative ergodic theorem (for a nice example, see theorem (1.6) in [32]). There exist many such theorems for various circumstances. The first multiplicative ergodic theorem was proven by Oseledec [33]; many others - [34], [35], [32], [36], [37], [38], and [25] - have subsequently generalized his original result. We will refrain from stating a specific multiplicative ergodic theorem; the conditions necessary for the existence of Lyapunov exponents are exactly the conditions we place on our function space in section in (IIC). In other words, a $C^r$ ($r > 0$) map of a compact manifold $M$ to itself and an $f$–invariant probability measure $\rho$, on $M$. For specific treatments we leave the curious reader to study the aforementioned references, noting that our construction follows from [35], [25], and [27].

There is an intimate relationship between Lyapunov exponents and global stable and unstable manifolds. In fact, each Lyapunov exponent corresponds to a global manifold. We will be using the global manifold structure as our measure of topological equivalence, and the Lyapunov exponents to classify this global structure. Positive Lyapunov exponents correspond to global unstable manifolds, and negative Lyapunov exponents correspond to global stable manifolds. We will again refrain from stating the existence theorems for these global manifolds, and instead note that in addition to the requirements for the existence of Lyapunov exponents, the existence of global stable/unstable manifolds corresponding the negative/positive Lyapunov exponents requires $Df$ to be injective. For specific global unstable/stable manifold theorems see [35].

The theories of hyperbolicity, Lyapunov exponents and structural stability have had a long, wonderful, and tangled history (for good starting points see [39] or [40]). We will, of course, not scratch the surface with our current discussion, but rather put forth the connections relevant for our work. Lyapunov exponents are the logarithmic average of the (properly normalized) eigenvalues of the local (linearization at a point) Jacobian along a given orbit. Thus for periodic orbits, the Lyapunov exponents are simply the log of the eigenvalues. A periodic orbit with period $p$ is hyperbolic if either the eigenvalues of the time $p$ map are not one, or the Lyapunov exponents are not zero. The connection between structural stability and hyperbolicity is quite beautiful and has a long and wonderful history beginning with Palis and Smale [41]. For purposes of interpretation later, it will be useful to state the solution of the stability conjecture:

**Theorem 2 (Mañé [4] theorem A, Robbin [2], Robinson [42])** A $C^1$ diffeomorphism (on a compact, boundaryless manifold) is structurally stable if and only if it satisfies axiom A and the strong transversality condition.

Recall that axiom A says the diffeomorphism is hyperbolic with dense periodic points on its non-wandering set $\Omega$ ($\rho \in \Omega$ is non-wandering if for any neighborhood $U$ of $x$, there is an $n > 0$ such that $f^n(U) \cap U \neq \emptyset$). We will save a further explicit discussion of this interrelationship for a later section, noting that much of this report investigates the above notions and how they apply to our set of maps.

Finally, for a nice, sophisticated introduction to the above topics see [30].

**C. Conditions needed for the existence of Lyapunov exponents**

Lyapunov exponents are one of our principal diagnostics, thus we must briefly justify their existence for our construction. We will begin with a standard construction for the existence and computation of Lyapunov exponents as defined by the theories of Katok [34], Ruelle [35], [32], Pesin [36], [37], [38], Brin and Pesin [25], and Burns, Dolgopyat and Pesin [27]. We will then note how this applies to our construction. (For more practical approaches to the numerical calculation of Lyapunov spectra see [43], [44], [45], and [46].)

Let $H$ be a separable real Hilbert space (for practical purposes $R^n$), and let $X$ be an open subset of $H$. Next let
(X, Σ, ρ) be a probability space where Σ is a σ-algebra of sets, and ρ is a probability measure, ρ(X) = 1 (see [47] for more information). Now consider a $C^r$ ($r > 1$) map $f_t : X \mapsto X$ which preserves ρ ($ρ$ is $f$-invariant) defined for $t \geq T_0 \geq 0$ such that $f_{t+s} = f_t \circ f_s$ and that $(x, t) \mapsto f_t(x)$, $Df_t(x)$ is continuous from $X \times [T_0, \infty)$ to $X$ and bounded on $\mathcal{H}$. Assume that $f$ has a compact invariant set

$$A = \{ t > T_0 \}
\inter\{ f_t(X)|f_t(A) \subseteq A \}$$

and $Df_t$ is a compact bounded operator for $x \in A$, $t > T_0$.

Finally, endow $f_t$ with a scalar parameter $s \in [0 : \infty]$. This gives us the space (a metric space - the metric will be defined heuristically in section II A 4) of one parameter, $C^r$ measure-preserving maps from bounded compact sets to themselves with bounded first derivatives. It is for a space of the above mappings that Ruelle shows the existence of Lyapunov exponents [35] (similar, requirements are made by Brin and Pesin [25] in a slightly more general setting).

Now we must quickly justify our use of Lyapunov exponents. Clearly, we can take $X$ in the above construction to be the $\mathbb{R}^d$ of section (II A 1). As our neural networks map their domains to compact sets, and they are constructed as time-delays, their domains are also compact.

Further, their derivatives are bounded up to arbitrary order, although for our purposes, only the first order need be bounded. Because the neural networks are deterministic and bounded, there will exist an invariant set of some type. All we need yet deal with is the measure preservation of which previously there is no mention. This issue is partially addressed in [48] in our neural network context. There remains much work to achieve a full understanding of Lyapunov exponents for general dissipative dynamical systems that are not absolutely continuous, for a current treatment see [29]. The specific measure theoretic properties of our networks (i.e. issues such as absolute continuity, uniform/non-uniform hyperbolicity, basin structures, etc) is a topic of current investigation.

**D. Definitions for numerical arguments**

Since we are conducting a numerical experiment, we will present some notions needed to test our conjectures numerically. We will begin with a notion of continuity. The heart of continuity is based on the following idea: if a neighborhood about a point in the domain is shrunk, this implies a shrinking of a neighborhood of the range. However, we do not have infinitesimals at our disposal. Thus, our statements of numerical continuity will necessarily have a statement regarding the limits of numerical resolution below which our results are uncertain.

Let us now begin with a definition of bounds on the domain and range:

**Definition 9** ($\epsilon_{num}$) $\epsilon_{num}$ is the numerical accuracy of a Lyapunov exponent, $\chi_j$.

**Definition 10** ($\delta_{num}$) $\delta_{num}$ is the numerical accuracy of a given parameter under variation.

Now, with our $\epsilon_{num}$ and $\delta_{num}$ defined as our numerical limits in precision, let us define numerical continuity of Lyapunov exponents.

**Definition 11** ($\text{num-continuous Lyapunov exponents}$) Given a one parameter map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $f \in C^r$, $r > 0$, for which characteristic exponents $\chi_j$ exist (and are the same under all invariant measures). The map $f$ is said to have num-continuous Lyapunov exponents at $(\mu, x) \in \mathbb{R}^d \times \mathbb{R}^d$ if for $\epsilon_{num} > 0$ there exists a $\delta_{num} > 0$ such that:

$$|s - s'| < \delta_{num}$$

then

$$|\chi_j(s) - \chi_j(s')| < \epsilon_{num}$$

for $s, s' \in \mathbb{R}^d$, for all $j \in \mathbb{N}$ such that $0 < j \leq d$.

Another useful definition related to continuity is that of a function being Lipschitz continuous.

**Definition 12** ($\text{num-Lipschitz}$) Given a one parameter map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $f \in C^r$, $r > 0$, for which characteristic exponents $\chi_j$ exist (and are the same under all invariant measures), the map $f$ is said to have num-Lipschitz Lyapunov exponents at $(\mu, x) \in \mathbb{R}^d \times \mathbb{R}^d$ if there exists a real constant $0 < k_{\chi_j}$ such that

$$|\chi_j(s) - \chi_j(s')| < k_{\chi_j}|s - s'|$$

Further, if the constant $k_{\chi_j} < 1$, the Lyapunov exponent is said to be contracting [66] on the interval $[s, s']$ for all $s'$ such that $|s - s'| < \delta_{num}$.

Note that neither of these definitions imply strict continuity, but rather, they provide bounds on the difference between the change in parameter and the change in Lyapunov exponents. It is important to note that these notions are highly localized with respect to the domain in consideration. We will not imply some sort of global continuity using the above definitions, rather, we will use these notions to imply that Lyapunov exponents will continuously (within numerical resolution) cross through zero upon parameter variation. We can never numerically prove that Lyapunov exponents don’t jump across zero, but for most computational exercises, a jump across zero that is below numerical precision is not relevant. This notion of continuity will aid in arguments regarding the existence of periodic windows in parameter space.

Let us next define a Lyapunov exponent zero-crossing:

**Definition 13** (Lyapunov exponent zero-crossing) A Lyapunov exponent zero-crossing is simply the point $s_{\chi_j}$ in parameter space such that a Lyapunov exponent continuously (or num-continuously) crosses zero. e.g. for $s - \delta$, $\chi_i > 0$, and for $s + \delta$, $\chi_i < 0$.
For this report, a Lyapunov exponent zero-crossing is a transverse intersection with the real line. For our networks non-transversal intersections of the Lyapunov exponents with the real line certainly occur, but for the portion of parameter space we are investigating, they are extremely rare. Along the route-to-chaos for our networks, such non-transversal intersections are common, but will save the discussion of that topic for a different report. Orbits for which the Lyapunov spectrum can be defined (in a numerical sense, Lyapunov exponents are defined when they are convergent), yet at least one of the exponents is zero are called non-trivially num—partially hyperbolic. We must be careful making statements with respect to the existence zero Lyapunov exponents implying the existence of corresponding center manifolds $E^c$ as we can do with the positive and negative exponents and their respective stable and unstable manifolds.

Lastly, we define a notion of denseness for a numerical context. There are several ways of achieving such a notion — we will use the notion of a dense sequence.

**Definition 14 ($\epsilon$-dense) Given an $\epsilon > 0$, an open interval $(a, b) \subset R$, and a sequence $\{c_1, \ldots, c_n\}$, $\{c_1, \ldots, c_n\}$ is $\epsilon$-dense in $(a, b)$ if there exists an $n$ such that for any $x \in (a, b)$, there is an $i$, $1 \leq i < n$, such that $dist(x, c_i) < \epsilon$.**

In reality however, we will be interested in a sequence of sequences that are “increasingly” $\epsilon$-dense in an interval $(a, b)$. In other words, for the sequence of sequences

$$c_1^1, \ldots, c_1^n, c_2^1, \ldots, c_2^n, \ldots,$$

where $n_{i+1} > n_i$ (i.e. for a sequence of sequences with increasing cardinality), the subsequent sequences for increasing $n_i$ become a closer approximation of an $\epsilon$-dense sequence. Formally —

**Definition 15 (Asymptotically Dense (a—dense)) A sequence $S_j = \{c_1^j, \ldots, c_n^j\} \subset (a, b)$ of finite subsets is asymptotically dense in $(a, b)$, if for any $\epsilon > 0$, there is an $N$ such that $S_j$ is $\epsilon$-dense if $j \geq N$.**

For an intuitive example of this, consider a sequence $S$ of $k$ numbers where $q_k \in S$, $q_k \in (0, 1)$. Now increase the cardinality of the set, spreading elements in such a way that they are uniformly distributed over the interval. Density is achieved with the cardinality of infinity, but clearly, with a finite but arbitrarily high number of elements, we can achieve any approximation to a dense set that we wish. There are, of course, many ways we can have a countably infinite set that is not dense, and, as we are working with numerics, we must concern ourselves with how we will approach this asymptotic density. We now need a clear understanding of when this definition will apply to a given set. There are many pitfalls; for instance, we wish to avoid sequences such as $(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$. We will, in the section that addresses a—density, state the necessary conditions for an a—dense set for our purposes.

### III. Conjectures

The point of this exercise is verifying three properties of $C^1$ maps along a one-dimensional interval in parameter space. The first property is the existence of a collection of points along an interval in parameter space such that hyperbolicity of the mapping is violated. The second property, which is really dependent upon the first and third properties, is the existence of an interval in parameter space of positive measure such that topological change (in the sense of changing numbers of unstable manifolds) with respect to slight parameter variation on the aforementioned interval is common. The final property we wish to show, which will be crucial for arguing the second property, is that on the aforementioned interval in parameter space, the topological change will not yield periodic windows in the interval if the dimension of the mapping is sufficiently high. More specifically, we will show that the ratio of periodic window size to parameter variation size ($\delta$) goes to zero on our chosen interval.

**Condition 1 Given a map (neural network) as defined in section (IIA 4), if the parameter $s \in R^1$ is varied num—continuously, then the Lyapunov exponents vary num—continuously.**

There are many counterexamples to this condition, so many of our results will rest upon our ability to show how generally the above condition applies in high-dimensional systems.

**Definition 16 (Chain link set) Assume $f$ is a mapping (neural network) as defined in section (IIA 4). A chain link set is denoted —**

$$V = \{s \in R \mid \chi_j(s) \neq 0 \text{ for all } 0 < j \leq d \text{ and } \chi_j(s) > 0 \text{ for some } j > 0\}$$

If $\chi_j(s)$ is continuous at its Lyapunov exponent zero-crossing, as we will show later (a la condition (1)), then $V$ is open. Next, let $C_k$ be a connected component of the closure of $V$, $\overline{V}$. It can be shown that $C_k \cap V$ is a union of disjoint, adjacent open intervals of the form $\bigcup_i (a_i, a_{i+1})$.

**Definition 17 (Bifurcation link set) Assume $f$ is a mapping (neural network) as defined in section (IIA 4). Denote a bifurcation link set of $C_k \cap V$ as:**

$$V_i = (a_i, a_{i+1})$$

(27)

Assume the number of positive Lyapunov exponents for each $V_i \subset V$ remains constant, if, upon a monotonically increasing variation in the parameter $s$, the number of
positive Lyapunov for $V_i$ is greater than the number of positive Lyapunov exponents for $V_{i+1}$, $V$ is said to be LCE decreasing. Specifically, the endpoints of $V_i$'s are the points where there exist Lyapunov exponent zero crossings. We are not particularly interested in these sets however, rather we are interested in the collection of endpoints adjoining these sets.

**Definition 18 (Bifurcation chain subset)** Let $V$ be a chain link set, and $C_k$ a connected component of $\nabla$. A 
**bifurcation chain subset** of $C_k \cap V$ is denoted:

$$U_k = \{a_i\} \quad (28)$$

or equivalently:

$$U_k = \partial(C_k \cap V) \quad (29)$$

For our purposes in this work, we will consider a bifurcation chain subset $U$ such that $a_1$ corresponds to the last zero crossing of the least positive exponent and $b_n$ will depend upon the specific case and dimension. In a practical sense, $a_1 \sim 0.5$ and $b_n \sim 6$. For higher dimensional networks, $b_n \sim 6$ will correspond to a much higher $n$ than for a low-dimensional network. For an intuitive picture of what we wish to depict with the above definitions, consider figure (2).

We will now state the conjectures, followed by some definitions and an outline of what we will test and why those tests will verify our claims.

**Conjecture 1 (Hyperbolicity violation)** Assume $f$ is a mapping (neural network) as defined in section (IIA 4) with a sufficiently high number of dimensions, $d$. There exists at least one bifurcation chain subset $U$.

The intuition arises from a straightforward consideration of the neural network construction in section (IIA 4). From consideration of our specific neural networks and their activation function, tanh(), it is clear that variation of the scaling parameter, $s$, on the variance of the interaction weights $\omega$ forces the neural networks from a linear region, through a non-linear region, and into a binary region. This implies that, given a neural network that is chaotic for some value of $s$, upon the monotonically increasing variation of $s$ from zero, the dynamical behavior will begin at a fixed point, proceed through a sequence of bifurcations, become chaotic, and eventually become periodic. If the number of positive Lyapunov exponents can be shown to increase with the dimension of the network and if the Lyapunov exponents can be shown to vary relatively continuously with respect to parameter variation with increasing dimension, then there will be many points along the parameterized curve such that there will exist neutral directions. The ideas listed above provide the framework for computational verification of conjecture (2). We must investigate conjecture (1) with respect to the subset $U$ becoming a dense in its closure and the existence of very few (ideally a single) connected components of $\nabla$.

**Conjecture 2 (Existence of a Codimension $\epsilon$ bifurcation set)** Assume $f$ is a mapping (neural network) as defined in section (IIA 4) with a sufficiently high number of dimensions, $d$, and a bifurcation chain set $U$ as per conjecture (1). The two following (equivalent) statements hold:

i. In the infinite-dimensional limit, the cardinality of $U$ will go to infinity, and the length $\max |a_{i+1} - a_i|$ for all $i$ will tend to zero on a one dimensional interval in parameter space. In other words, the bifurcation chain set $U$ will be a dense in its closure, $\nabla$.

ii. In the asymptotic limit of high dimension, for all $s \in U$, and for all $f$ at $s$, an arbitrarily small perturbation $\delta_s$ of $s$ will produce a topological change. The topological change will correspond to a different number of global stable and unstable manifolds for $f$ at $s$ compared to $f$ at $s + \delta$.

Assume $M$ is a $C^r$ manifold of topological dimension $d$ and $N$ is a submanifold of $M$. The codimension of $N$ in $M$ is defined as $\text{codim}(N) = \dim(M) - \dim(N)$. If there exists a curve $p$ through $M$ such that $p$ is transverse to $N$ and the $\text{codim}(N) \leq 1$, then there will not exist an arbitrarily small perturbation of $p$ such that $p$ will become non-transverse to $N$. Moreover, if $\text{codim}(N) = 0$ and $p \cap N \subset \text{int}(N)$, then there does not even exist an arbitrarily small perturbation of $p$ such that $p$ intersects $N$ at a single point of $N$, i.e. the intersection cannot be made non-transverse with an arbitrarily small perturbation.

The former paragraph can be more easily understood via figure (3) where we have drawn four different circumstances. This first circumstance, the curve $p_1 \cap N$, is an example of a non-transversal intersection with a codimension 0 submanifold. This intersection can be perturbed away with an arbitrarily small perturbation of $p_1$. The intersection, $p_2 \cap N$, is a transversal intersection with a codimension 0 submanifold, and this intersection cannot be perturbed away with an arbitrarily small perturbation of $p_2$. Likewise, the intersection, $p_1 \cap O$, which is an example of a transversal intersection with a codimension 1 submanifold cannot be made non-transverse or null via an arbitrarily small perturbation of $p_1$. The intersection $p_2 \cap O$ is a non-transversal intersection with a codimension 1 submanifold and can be perturbed away with an arbitrarily small perturbation of $p_2$. This outlines the avoid-ability of codimension 0 and 1 submanifolds with respect to curves through the ambient manifold $M$. The point is that non-null, transversal intersections of curves with codimension 0 or 1 submanifolds cannot be made non-transversal with arbitrarily small perturbations of the curve. Transversal intersections of curves with codimension 2 submanifolds, however, can always be removed by an arbitrarily small perturbation due to the existence of a “free” dimension. A practical example of such would be the intersection of a curve with another curve in $\mathbb{R}^3$ — one can always pull apart the two curves simply by ‘lifting’ them apart.
FIG. 2: An intuitive diagram for chain link sets, $V$, bifurcation link sets, $V_i$, and bifurcation chain sets, $U$. For an LCE decreasing chain link set $V$.

FIG. 3: The top drawing represents various standard pictures from transversality theory. The bottom drawing represents an idealized version (in higher dimensions) of transversality catering to our arguments.

In the circumstance proposed in conjecture (2), the set $U$ ($\tilde{N}$ in the Fig. (3)) will always have codimension $d$ because $U$ consists of finitely many points, thus any intersection with $U$ can be removed by an arbitrarily small perturbation. The point is that, as $U$ becomes $a$-dense in $\tilde{U}$, $p_3 \cap \tilde{U} = 0$ becomes more and more unlikely and the perturbations required to remove the intersections of $p_3$ with $U$ (again, $\tilde{N}$ as in the Fig. (3)) will become more and more bizarre. For a low-dimensional example, think of a ball of radius $r$ in $\mathbb{R}^3$ that is populated by a finite set of evenly distributed points, denoted $S_i$, where $i$ is the number of elements in $S_i$. Next fit a curve $p$ through that ball in such a way that $p$ does not hit any points in $S_i$. Now, as the cardinality of $S_i$ becomes large, if $S_i$ is $a$-dense in the ball of radius $r$, for the intersection of $p$ with $S_i$ to remain null, the $p$ will need to become increasingly kinky. Moreover, continuous, linear transformations of $p$ will become increasingly unlikely to preserve $p \cap S_i = 0$. It is this type of behavior with respect to parameter variation that we are arguing for with conjecture (2). However, figure (3) is should only be used as an tool for intuition — our conjectures are with respect to a particular interval in parameter space and not a general curve in parameter space, let alone a family of curves or a high-dimensional surface. Conjecture (2) is a first step towards a more complete argument with respect to the above scenario. For more information for where the above picture originates, see [49] or [50].

To understand roughly why we believe conjecture (2) is reasonable, first take condition (1) for granted (we will expend some effort showing where condition (1) is reasonable). Next assume there are arbitrarily many Lyapunov exponents near 0 along some interval of parameter space and that the Lyapunov exponent zero-crossings can be shown to be $a$-dense with increasing dimension. Further, assume that on the aforementioned interval, $V$ is LCE decreasing. Since varying the parameters continuously on some small interval will move Lyapunov exponents continuously, small changes in the parameters will guarantee a continual change in the number of positive Lyapunov exponents. One might think of this intuitively relative to the parameter space as the set of Lyapunov exponent zero-crossings forming a codimension 0 submani-
fold with respect to the particular interval of parameter space. However, we will never achieve such a situation in a rigorous way. Rather, we will have an $a$-dense bifurcation chain set $U$, which will have codimension 1 in $R$ with respect to topological dimension. As the dimension of $f$ is increased, $U$ will behave more like a codimension 0 submanifold of $R$. Hence the metaphoric language, codimension $\epsilon$ bifurcation set. The set $U$ will always be a codimension one submanifold as it is a finite set of points. Nevertheless, if $U$ tends toward being dense in its closure, it will behave increasingly like a codimension zero submanifold. This argument will not work for the entirety of the parameter space, and thus we will show where, to what extent, and under what conditions $U$ exists and how it behaves as the dimension of the network is increased.

Conjecture 3 (Periodic window probability decreasing) Assume $f$ is a mapping (neural network) as defined in section (II A 4) and a bifurcation chain set $U$ as per conjecture (1). In the asymptotic limit of high dimension, the length of the bifurcation chain sets, $l = |a_n - a_1|$, increases such that the cardinality of $U \rightarrow m$ where $m$ is the maximum number of positive Lyapunov exponents for $f$. In other words, there will exist an interval in parameter space (e.g. $s \in (a_1, a_n) \sim (0.1, 4)$) where the probability of the existence of a periodic window will go to zero (with respect to Lebesgue measure on the interval) as the dimension becomes large.

This conjecture is somewhat difficult to test for a specific function since adding inputs completely changes the function. Thus the curve through the function space is an abstraction we are not afforded by our construction. We will save a more complete analysis (e.g. a search for periodic windows along a high-dimensional surface in parameter space) of conjecture (3) for a different report. In this work, conjecture (3) addresses a very practical matter, for it implies the existence of a much smaller number of bifurcation chain sets. The previous conjectures allow for the existence of many of these bifurcation chains sets, $U$, separated by windows of periodicity in parameter space. However, if these windows of periodic dynamics in parameter space vanish, we could end up with only one bifurcation chain set — the ideal situation for our arguments. We will not claim such, however we will claim that the length of the set $U$ we are concerning ourselves with in a practical sense will increase with increasing dimension, largely due to the disappearance of periodic windows on the closure of $V$. With respect to this report, all that needs be shown is that the window sizes along the path in parameter space for a variety of neural networks decreases with increasing dimension. From a qualitative analysis it will be somewhat clear that the above conjecture is reasonable.

If this were actually making statements we could rigorously prove, conjectures (1), (2), and (3) would function as lemmas for conjecture (4).

Conjecture 4 Assume $f$ is a mapping (neural network) as defined in section (II A 4) with a sufficiently high number of dimensions, $d$, a bifurcation chain set $U$ as per conjecture (1), and the chain link set $V$. The perturbation size $\delta$ of $s \in C_{max}$, where $C_{max}$ is the largest connected component of $V$, for which $f|_{C_k}$ remains structurally stable goes to zero as $d \rightarrow \infty$.

Specific cases and the lack of density of structural stability in certain sets of dynamical systems has been proven long ago. These examples were, however, very specialized and carefully constructed circumstances and do not speak to the commonality of structural stability failure. Along the road to investigating conjecture (4) we will show that structural stability will not, in a practical sense, be observable for a large set of very high-dimensional dynamical systems along certain, important intervals in parameter space even though structural stability is a property that will exist on that interval with probability one (with respect to Lebesgue measure). To some, this conjecture might appear to contradict some well-known results in stability theory. A careful analysis of this conjecture, and its relation to known results will be discussed in sections (VII A 4) and (VII C 1).

The larger question that remains, however, is whether conjecture (4) is valid on high-dimensional, bounded, nonlinear dynamical systems. We believe this is a much more difficult question with a much more complicated answer. We can, however, speak to a highly related problem, the problem of whether chaos persists in high-dimensional dynamical systems. Thus, let us now make a very imprecise conjecture that we will make more concise in a later section.

Conjecture 5 Chaos is a robust, high-probability behavior for high-dimensional, bounded, nonlinear dynamical systems.

This is not a revelation (as previously mentioned, many experimentalists have been attempting to break this robust, chaotic behavior for the last hundred years), nor is it a particularly precise statement. We have studied this question using neural networks much like those described in section (II A 4), and we found that for high-dimensional networks with a sufficient degree of nonlinearity, the probability of chaos was near unity [51]. Over the course of investigation of the above claims, we will see a qualitative verification of chaos was near unity [51]. Over the course of investigation of the above claims, we will see a qualitative verification of chaos was near unity [51]. Over the course of investigation of the above claims, we will see a qualitative verification of chaos was near unity [51]. Over the course of investigation of the above claims, we will see a qualitative verification of chaos was near unity [51].

IV. NUMERICAL ERRORS IN LYAPUNOV EXPONENT CALCULATION

Before we commence with our numerical arguments for the above conjectures, we analyze the numerical errors for both insight into how our chief diagnostic works and
to establish bounds of accuracy on the numerical results that will follow. We will proceed first with an analysis of single networks of varying dimensions, providing intuition into the evolution of the calculation of the Lyapunov spectrum versus iteration time. We will follow this analysis with a statistical study of 1000 networks, measuring the deviation from the mean of the exponent over 10000 time steps, thus noting how the individual exponents converge and to what extent the exponents of all the networks converge.

We will begin by considering Fig. (4), plots of the Lyapunov spectrum versus the first 10000 iterations for two networks with 16 and 64 dimensions. After approximately 3000 time steps, all the large transients have essentially vanished, and aside from slight variation (especially on a time scale long compared with a single timestep) the exponents appear to have converged. For the case with 16 dimensions the exponents also appear to have converged. The resolution for the network with 64 dimensions is not fine enough to verify a distinction between exponents, thus consideration of Fig. (5) demonstrates clearly that the exponents converge well within the inherent errors in the calculation, and are entirely distinct for time steps greater than 5500 time steps. It is worth noting that there are times when very long term transients occur in our networks. These transients would not be detectable from the figures we have presented, but these problem cases usually only exist near bifurcation points. For the cases we are considering, these convergence issues do not seem to affect our results[67].

Figures (4) and (5) provide insight into how the individual exponents for individual networks converge; we now must establish the convergence of the Lyapunov exponents for a large set of neural networks and present a general idea of the numerical variance ($\epsilon_m$) in the Lyapunov exponents. We will achieve this in the following manner: we will calculate the Lyapunov spectrum for an individual network for 5000 time steps; we will calculate the mean of each exponent in the spectrum; we will, for each time step calculate the deviation of the exponent from the mean of that exponent; we will follow the above procedure for 1000 networks and take the mean of the deviation from the mean exponent at each time step. Figure (6) represents the analysis in the former statement. This figure demonstrates clearly that the deviation from the mean exponent, even for the most negative exponent (the most negative exponent has the largest error) drops below 0.01 after 3000 time steps. The fluctuations in the largest Lyapunov exponent lie in the $10^{-3}$ range for 3000 time-steps. Figure (6) also substantiates three notions: a measurement of how little the average exponent strays from its mean value; a measurement of the similarity of this characteristic over the ensemble of networks; and finally it helps establish a general intuition with respect to the accuracy of our exponents, $\epsilon_m < 0.01$ for 5000 time steps.

It is worth noting that determining errors in the Lyapunov exponents is not an exact science; for our networks such errors vary a great deal in different regions in $s$ space. For instance, near the first bifurcation from a fixed point can require up to 100000 or more iterations to converge to an attractor and 50000 more iterations for the Lyapunov spectrum to converge.

\section{V. NUMERICAL ARGUMENTS FOR PRELIMINARIES}

Before we present our arguments supporting our conjectures we must present various preliminary results. Specifically we will discuss the $num-$continuity of the Lyapunov exponents, the $a-$density of Lyapunov exponent zero-crossings, and argue for the existence of arbitrarily high number of positive exponents given an arbitrarily high number of dimensions. With these preliminaries in place, the arguments supporting our conjectures will be far more clear.

\subsection{A. $num-$continuity}

Testing for the $num-$continuity of Lyapunov exponents formally will be two-fold. First, we will need to investigate, for a specific network, $f$, the behavior of Lyapunov exponents versus variation of parameters. Second, indirect, yet strong evidence of the $num-$continuity will also come from investigating how periodic window size varies with dimension and parameter variation. It is important to note that when we refer to continuity, we are referring to a very local notion of continuity. Continuity is always in reference to the set upon which something (a function, a mapping, etc) is continuous. In the below analysis, the neighborhoods upon which continuity of the Lyapunov exponents are examined is over ranges of plus and minus one parameter increment. This is all that is necessary for our purposes, but this analysis cannot guarantee strict continuity along, say, $s \in [0.1, 10]$, but rather continuity along little linked bits of the interval $[0.1, 10]$.

\subsubsection{1. Qualitative analysis}

Qualitatively, our intuition for $num-$continuity comes from examining hundreds of Lyapunov spectrum plots versus parameter variation. In this vein, Figs. (7) and (8) present the difference between low and higher dimensional Lyapunov spectra.

In Fig. (8), the Lyapunov exponents look continuous within numerical errors (usually $\pm 0.005$). Figure (8) by itself provides little more than an intuitive picture of what we are attempting to argue. As we will be making arguments that the Lyapunov spectrum will become more smooth, periodic windows will disappear, etc, with increasing dimension, Fig. (7) shows a typical graph of the Lyapunov spectrum versus parameter variation for a neural network with 32 neurons and 4 dimensions. The
contrast between Figs. (8) and (7) intuitively demonstrates the increase in continuity we are claiming.

Although a consideration of Figs. (7) and (8) yields an observation that, as the dimension is increased, the Lyapunov exponents appear to be more continuous function of the \( s \) parameter, the above figures alone do not verify \( \text{num-continuity} \). In fact, it should be noted that pathological discontinuities have been observed in networks with as many as 32 dimensions. The existence of pathologies for higher dimensions is not a problem we are prepared to answer in depth; it can be confidently said that as the dimension (number of inputs) is increased, the frequency of pathologies appears to become vanishingly rare (this is noted over our observation of several thousand networks with dimensions ranging from 4 to 256).

2. Quantitative and numerical analysis

Our quantitative analysis will follow two lines. The first will be a specific analysis along the region of parameter change for three networks with dimensions 4 and 64, respectively. This will be followed with a more statistical study of a number of networks per dimension where the dimensions will range from 4 to 128 in powers of 2.

Consider the \( \text{num-continuity} \) of two different networks while varying the \( s \) parameter. Figure (9) is a plot of the mean difference in each exponent between parameter values summed over all the exponents. The parameter increment is \( \delta s = 0.01 \).

The region of particular interest is between \( s = 0 \) and 6. Considering this range, it is clear that the variation in the mean of the exponents versus variation in \( s \) decreases with dimension. The 4-dimensional network not only has a higher baseline of \( \text{num-continuity} \), but it also has many large spikes. As the dimension is increased, considering the 64-dimensional case, the baseline of \( \text{num-continuity} \) is decreased, and the large spikes disappear. The spikes in the 4-dimensional case can be directly linked to the existence of periodic windows and bifurcations that result in dramatic topological change. This is one verification of \( \text{num-continuity} \) of Lyapunov exponents. These two cases are quite typical, but it is clear that the above analysis, although quite persuasive, is not adequate for our needs. We will thus resort to a statistical study of the above plots.

The statistical support we have for our claim of increased \( \text{num-continuity} \) will focus on the parameter region between \( s = 0 \) and 6, the region in parameter space over which the maxima of entropy, Kaplan-Yorke dimension, and the number of positive Lyapunov exponents exists. Figure (10) considers the \( \text{num-continuity} \) along parameter values ranging from 0 to 6. The points on the plot correspond to the mean (over a few hundred networks) of the mean exponent change between parameter values, or:

\[
\mu^d = \frac{1}{Z} \sum_{k=1}^{Z} \sum_{i=1}^{d} |\chi_k^i(s) - \chi_k^i(s + \delta s)|
\]

where \( Z \) is the total number of networks of a given dimension considered.

Figure (10) clearly shows that as the dimension is increased, for the same computation time, both the mean exponent change versus parameter variation per network and the standard deviation of the exponent change decrease substantially as the dimension is increased.[68] Of
course the mean change over all the exponents allows for the possibility for one exponent (possibly the largest exponent) to undergo a relatively large change while the other exponents change very little. For this reason, we have included the num—continuity of the largest and the most negative exponents versus parameter change. The num—continuity of the largest exponents is very good, displaying a small standard deviation across many networks. The error in the most negative exponent is inherent to our numerical techniques (specifically the Gram-Schmidt orthogonalization). The error in the most negative exponent increases with dimension, but is a numerical artifact. This figure yields strong evidence that in the region of parameter space where the network starts at a fixed point (all negative Lyapunov exponents), grows to having the maximum number of positive exponents, and returns to having a few positive exponents, the variation in any specific Lyapunov exponent is very small.

There is a specific relation between the above data to definition 12; num—Lipschitz is a stronger condition than num—continuity of Lyapunov exponents. The mean num—continuity at $n = 32, d = 4$

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.02] < k[0.01] \tag{31}

yielding $k = 2$ which would not classify as num—Lipschitz contracting, whereas for $n = 32, d = 128$ we arrive at

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.004] < k[0.01] \tag{32}

which yields $k = 0.4 < 1$ which does satisfy the condition for num—Lipschitz contraction. Even more striking is the num—continuity of only the largest Lyapunov exponent; for $n = 32, d = 4$ we get

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.015] < k[0.01] \tag{33}

which yields $k = 1.5$, while the $n = 32 d = 128$ case is

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.002] < k[0.01] \tag{34}

yielding $k = 1.5$ which does not classify as num—Lipschitz contracting, whereas for $n = 32, d = 128$ we arrive at

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.0015] < k[0.01] \tag{35}

which yields $k = 1.5$, while the $n = 32 d = 128$ case is

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.001] < k[0.01] \tag{36}

which yields $k = 1.5$, while the $n = 32 d = 128$ case is

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.0005] < k[0.01] \tag{37}

which yields $k = 1.5$, while the $n = 32 d = 128$ case is

$$|\chi_j(s + \delta_{num}) - \chi_j(s)| < k\delta_{num}$$

\phantom{|} [0.0002] < k[0.01] \tag{38}
which nets $k = 0.2$. As the dimension is increased, $k$ decreases, and thus num–continuity increases. As can be seen from Fig. (10), the num–continuity is achieved rather quickly as the dimension is increased; the Lyapunov exponents are quite continuous with respect to parameter variation by 16 dimensions. For an understanding in an asymptotic limit of high dimension, consider Fig. (11). As the dimension is increased the log of the dimension versus the log$_2(k_{\chi_1})$ yields the scaling $k \sim \sqrt{\frac{2}{d}}$; thus as $d \to \infty$, $k_{\chi_1} \to 0$, which is exactly what we desire for continuity in the Lyapunov exponents versus parameter change. This completes our evidence for the num–continuity in high-dimensional networks.

3. Relevance

Conjectures (1), (2), and (4) are all fundamentally based on condition (1). For the neural networks, all we need to establish conjecture (1) is the num–continuity of the Lyapunov exponents, the existence of the fixed point for $s$ near 0, the periodic orbits for $s \to \infty$, and three exponents that are, over some region of parameter variation, 32 neurons, 64 dimensions, $s \in [0, 11, 10]$.

FIG. 8: LE spectrum: 32 neurons, 64 dimensions, $s \in [0, 11, 10]$

FIG. 9: num–continuity (mean of $|\chi_i(s) - \chi_i(s + \delta s)|$ for each $i$) versus parameter variation: 32 neurons, 4 (left) and 64 (right) dimensions.

FIG. 10: Mean num–continuity, num–continuity of the largest and the most negative Lyapunov exponent of many networks versus their dimension. The error bars are the standard deviation about the mean over the number of networks considered.
ter space, all simultaneously positive. The n-continuity of Lyapunov exponents implies, within numerical precision, that Lyapunov exponents both pass through zero (and don’t jump from positive to negative without passing through zero) and are, within numerical precision, zero.

B. \(\alpha\)-density of zero crossings

Many of our arguments will revolve around varying \(s\) in a range of 0.1 to 6 and studying the behavior of the Lyapunov spectrum. One of the most important features of the Lyapunov spectrum we will need is a uniformity in the distribution of positive exponents between 0 and \(\chi_{\text{max}}\). As we are dealing with a countable set, we will refrain from any type of measure theoretic notions, and instead rely on \(\alpha\)-density of the set of positive exponents as the dimension is increased. Recall the definition of \(\alpha\)-dense (definition (15)), the definition of a bifurcation chain subset (definition (18)), which corresponds to the set of Lyapunov exponent zero crossings, and the definition of a chain link set (definition (16)). Our conjectures will make sense if and only if, as the dimension is increased, the bifurcation chain subsets become “increasingly” dense, or \(\alpha\)-dense in the closure of the chain link set (\(\bar{V}\)). The notion of \(\alpha\)-dense bifurcation chain set in the closure of the chain link set as dimension is increased that provides us with the convergence to density of non-hyperbolic points we need to satisfy our goals.

1. Qualitative analysis

The qualitative analysis will focus on pointing out what characteristics we are looking for and why we believe \(\alpha\)-density of Lyapunov exponent zero-crossings (\(\alpha\)-dense bifurcation chain set in the closure of the chain link set) over a particular region of parameter space exists. A byproduct of this analysis will be a picture of one of the key traits needed to support our conjectures. We will begin with figures showing the positive Lyapunov spectrum for 16 and 64 dimensions.

Considering the 16-dimensional case, and splitting the \(s\) parameter variation into two regions, region one - \(R_I = [0, 0.5]\), and region two - \(R_{II} = [0.5, 10]\). We then partition up \(R_{II}\) using the bifurcation link sets, and collect the zero crossings in the bifurcation chain sets.

We want the elements of the bifurcation chain sets to be spaced evenly enough so that, as the dimension goes to infinity, variations in the \(s\) parameter on the chain link set will lead to a Lyapunov exponent zero-crossing (and a transition from \(V_i\) to \(V_{i\pm1}\))[69]. Considering region \(II[70]\), we wish for the distance along the \(s\) axis between Lyapunov exponent zero-crossings (elements of the bifurcation chain subset) to decrease as the dimension is increased. If, as the dimension is increased, the Lyapunov exponents begin to “bunch-up” and cease to be at least somewhat uniformly distributed, the rest of our arguments will surely fail. For instance, in region two of the bottom plot of Fig. (12), if the Lyapunov exponents were “clumped,” there will be many holes where variation of \(s\) will not imply an exponent crossing. Luckily, considering the 64-dimensional case as given in Fig. (12), our desires seem to be as the spacing between exponent zero-crossings is clearly decreasing as the dimension is increased (consider the region \([0.5, 4]\)), and there are

![FIG. 11: \(k\)-scaling: \(\log_2\) of dimension versus \(\log_2\) of \(\alpha\)-Lipschitz constant of the largest Lyapunov exponent.](image1)

![FIG. 12: Positive LE spectrum for typical individual networks with 32 neurons and 16 (top) and 64 (bottom) dimensions.](image2)
FIG. 13: Number of positive LE’s for typical individual networks with 32 neurons and 32 (top) and 128 (bottom) dimensions.

no point accumulations of exponents. It is also reassuring to note that even at 16 dimensions, and especially at 64 dimensions, the Lyapunov exponents are quite distinct and look num−continuous as previously asserted. The above figures are, of course, only a picture of two networks; if we wish for a more conclusive statement, we will need arguments of a statistical nature.

2. Quantitative and numerical analysis

Our analysis that specifically targets the a−density of Lyapunov exponent zero crossings focuses on an analysis of plots of the number of positive exponents versus the s parameter.

Qualitatively, the two examples given in Fig. (13) (both of which typify the behavior for their respective number of neurons and dimensions) exemplify the a−density for which we are searching. As the dimension is increased, the plot of the variation in the number of positive exponents versus s becomes more smooth[71], while the width of the peak becomes more narrow. Thus, the slope of the number of positive exponents versus s between \( s = s_\ast \) (\( s_\ast \) is s where there exists the maximum number of positive Lyapunov exponents), and \( s = 2 \) drops from −3 at \( d = 32 \) to −13 at \( d = 128 \). Noting that the more negative the slope, the less variation in s is required to force a zero-crossing, it is clear that this implies

From Fig. (14), it is clear that as the dimension of the network is increased, the mean distance between successive exponent zero-crossings decreases. Note that measuring the mean distance between successive zero-crossings both in an intuitive and brute force manner, verifies the sufficient condition for the a−density of the set of s values for which there exist zero-crossings of exponents. The error bars represent the standard deviation of the length between zero-crossing over an ensemble (several hundred for low dimensions, on the order of a hundred for \( d = 128 \)) networks. For the cases where the dimension was 16 and 32, the s increment resolution was \( \delta s = 0.01 \). The error in the zero crossing distance for these cases is, at the smallest, 0.02, while at its smallest the zero crossing distance is 0.49, thus resolution of 0.01 in s variation is sufficient to adequately resolve the zero crossings. Such is not the case for 64 and 128 dimensional networks. For these cases we were required to increase the s resolution to 0.005. The zero crossings of a few hundred networks considered were all examined by hand; the distances between the zero crossings were always distinct, with a resolution well below that necessary to determine the zero crossing point. The errors were also determined by hand, noting the greatest, and least reasonable point for the zero crossing. All the zero crossings were determined after the smallest positive exponent that became positive hit its peak value, i.e. after approximately 0.75 in the \( d = 16 \) case of Fig. (12).
3. Relevance

The $a$–density of zero crossings of Lyapunov exponents provides the most important element in our arguments of conjectures (1) and (2); combining $num$–continuity with $a$–density will essentially net our desired results. If continuity of Lyapunov exponents increases, and the density of zero crossings of exponents increases over a set $U \in \mathbb{R}^1$ of parameter space, it seems clear that we will have both hyperbolicity violation and, upon variation of parameters in $U$, we will have the topological change we are claiming. Of course small issues remain, but those will be dealt with in the final arguments.

C. Arbitrarily large number of positive exponents

For our $a$–density arguments to work, we need a set whose cardinality is asymptotically a countably infinite set (such that it can be $a$–dense in itself) and we need the distance between the elements in the set to approach zero. The later characteristic was the subject of the previous section, the former subject is what we intend to address in this section.

1. Qualitative analysis

The qualitative analysis of this can be seen in Fig. (13); as the dimension is increased, the maximum number of positive Lyapunov exponents clearly increases. We wish to quantify that the increase in the number of positive exponents versus dimension occurs for a statistically relevant set of networks.

2. Quantitative analysis

We will use a brute force argument to demonstrate the increase in positive Lyapunov exponents with dimension; we will simply plot the number of positive exponents at the maximum number of exponents as dimension is increased. We claim that the number of Lyapunov exponents increases and, in fact, diverges to infinity as the limit dimension of the network is taken to infinity. Figure (15) showing the number of positive Lyapunov exponents versus dimension.

From Fig. (15) it is clear that as the dimension is increased, the number of positive exponents increases in a nearly linear fashion [72]. Further, this plot is linear to as high a dimension as the authors could compute enough cases for reasonable statistics. If the maximum number of exponents versus dimension remains linear beyond the range we could compute, we will have the countably infinite number of positive exponents we require.

FIG. 15: Mean maximum number of positive LE’s versus dimension, all networks have 32 neurons (slope is approximately $\frac{1}{4}$).

3. Relevance

The importance of the increasing number of positive exponents with dimension is quite simple. For the $a$–density of exponent zero crossing to be meaningful in the infinite-dimensional limit, there must also be an arbitrarily large number of positive exponents that can cross zero. If, asymptotically, there is a finite number of positive exponents, all of our claims will be false; $a$–density requires a countably infinity set.

VI. NUMERICAL ARGUMENTS FOR CONJECTURES

A. Decreasing window probability

With the $num$–continuity and $a$–density arguments already in place, all the evidence required to show the length of periodic windows along a curve in parameter space is already in place. We will present a bit of new data, but primarily we will clarify exactly what the conjecture says. We will also list the specifics under which the conjecture applies in our circumstances.

1. Qualitative analysis

Qualitative evidence for the dissappearance of periodic windows amidst chaos is evident from Figs. (7), (8) and (12); the periodic windows that dominate the 4-dimensional network over the parameter range $s = 0$ to 10 are totally absent in the 64-dimensional network. It is important to note that for this conjecture, as well as all our conjectures, we are considering the $s$ parameter over ranges no larger than 0 to 10. We will avoid, for the most part, the “route to chaos” region ($s$ near zero), as it
yields many complex issues that will be saved for another report. We will instead consider the parameter region after the lowest positive exponent first becomes positive. We could consider parameter ranges considerably larger, but for very large, the round-off error begins to play a significant role, and the networks become binary. This region has been briefly explored in [54]; further analysis is necessary for a more complete understanding [55].

2. Quantitative and numerical analysis

The quantitative analysis we wish to perform will involve arguments of two types; those that are derived from data given in sections (V A) and (V B), and those that follow from statistical data regarding the probability of a window existing for a given \( s \) along an interval in \( R \). We begin by recalling what we are attempting to claim and what conditions we need to verify the claim. We will then present the former argument and conclude with the latter.

The conjecture we are investigating claims that as the dimension of a dynamical system is increased, periodic windows along a one-dimensional curve in parameter space vanish in a significant portion of parameter space for which the dynamical system is chaotic. This is, of course, dependent upon the region of parameter space one is observing — and there is likely no way to rid ourselves of such an issue. For our purposes, we will generally be investigating the region of \( s \) parameter space between 0.1 and 10, however, sometimes we will limit the investigation to \( s \) between 2 and 4. Little changes if we increase \( s \) until the network begins behaving as a binary system due (quite possibly) to the round-off error. However, along the transition to the binary region, there are significant complications which we will not address here. As the dimension is increased, the main concern is that the lengths of the bifurcation chain sets must increase such that there will exist at least one bifurcation chain set that has a cardinality approaching infinity as the dimension of the network approaches infinity.

Our first argument is based directly upon the evidence of \textit{num}—continuity of Lyapunov exponents. From Fig. (10) it is clear that as the dimension of the set of networks sampled is increased, the mean difference in Lyapunov exponents over small (\( \delta s = 0.01 \)) \( s \) parameter perturbation decreases. This increase in \textit{num}—continuity of the Lyapunov exponents with dimension over our parameter range is a direct result of the disappearance of periodic windows from the chaotic regions of parameter space. This evidence is amplified by the decrease in the standard deviation of the \textit{num}—continuity versus dimension (of both the mean of the exponents and the largest exponent). This decrease in the standard deviation of the \textit{num}—continuity of the largest Lyapunov exponent allows for the existence of fewer large deviations in Lyapunov exponents (large deviations are needed for all the exponents to suddenly become less than or equal to zero).

We could consider parameter ranges considerably larger, but for very large, the round-off error begins to play a significant role, and the networks become binary. This region has been briefly explored in [54]; further analysis is necessary for a more complete understanding [55].

![FIG. 16: log\(_2\) of the probability of periodic or quasi-periodic windows versus log\(_2\) of dimension. The line \( P_w = 2.16d^{-1.128} \) is the least squares fit of the plotted data.](image)

Decreasing window probability inside the chaotic region provides direct evidence for conjectures (3) and (5) along a one-dimensional interval in parameter space. We will, in a more complete manner, attack those conjectures in a different report. We will use the decreasing periodic window probability to help verify conjecture (2) since it provides the context we desire with the \textit{num}—continuity of the Lyapunov spectrum. Our argument requires that there exists at least one maximum in the number of positive Lyapunov exponents with parameter variation. Further, that maximum must increase monotonically with the dimension of the system. The existence of periodic windows causes the following problems: periodic windows can still yield structural instability - but in a catastrophic way; periodic windows split up our bifurcation chain sets which, despite not being terminal to our arguments, provide many complications with which we do not contend. However, we do observe a decrease in pe-
periodic windows and with the decrease in the (numerical) existence of periodic windows comes the decrease in the number of bifurcation chain sets; i.e. \( l = |b_n - a_1| \) is increasing yet will remain finite.

B. Hyperbolicity violation

We will present two arguments for hyperbolicity violation - or nearness to hyperbolicity violation of a map at a particular parameter value, \( s \). The first argument will consider the fraction of Lyapunov exponents near zero over an ensemble of networks versus variation in the \( s \) parameter. If there is any hope of the existence of a chain link set with bifurcation link sets of decreasing length, our networks (on the \( s \) interval in question) must always have a Lyapunov exponent near zero. The second argument will come implicitly from \( a \)–density arguments presented in section (V B). To argue for this conjecture, we only need the existence of a neutral direction\[73\], or, more accurately, at least two bifurcation link sets, which is not beyond reach.

1. Qualitative analysis

A qualitative analysis of hyperbolicity violation comes from combining the \textit{num}–continuity of the exponents in Fig. (8) and the evidence of exponent zero crossings from Figs. (13) and (10). If the exponents are continuous with respect to parameter variation (at least locally) and they start negative, become positive, and eventually become negative, then they must be zero (within numerical precision) for at least two points in the parameter space. It happens that the bifurcation chain link sets are LCE decreasing from \( i \) to \( i + 1 \), which will provide additional, helpful, structure.

2. Quantitative and numerical analysis

The first argument, which is more of a necessary but not sufficient condition for the existence of hyperbolicity violation, consists of searching for the existence of Lyapunov exponents that are zero within allowed numerical errors. With \textit{num}–continuity, this establishes the existence of exponents that are numerically zero. For an intuitive feel for what numerically zero means, consider the oscillations in Fig. (13) of the number of positive exponents versus parameter variation. It is clear that as they cross zero there are numerical errors that cause an apparent oscillation in the exponent; these oscillations are due largely to numerical fluctuations in the calculations[74]. There is a certain fuzziness in numerical results that is impossible to remove, thus questions regarding exponents being exactly zero are ill-formed. Numerical results of the type presented in this paper need to be viewed in a framework similar to physical experimental results. With this in mind, we need to note the significance of the exponents near zero. To do this, we calculate the relative number of Lyapunov exponents numerically at zero compared to the ones away from zero. All this information can be summarized in Fig. (17) which addresses the mean fraction of exponents that are near zero versus parameter variation.

The cut-off for an exponent being near zero is \( \pm 0.01 \), which is approximately the expected numerical error in the exponents for the number of iterations we are using. There are four important features to notice about Fig. (17): there are no sharp discontinuities in the curves; there exists an interval in parameter space such that there is always at least one Lyapunov exponent in the interval \((-0.01, 0.01)\) and the length of that parameter interval is increasing with dimension; the curves are concave — implying that exponents are somehow leaving the interval \((-0.01, 0.01)\); and there is a higher fraction of exponents near zero at the same \( s \) value for higher dimension. The first property is important because holes in the parameter space where there are no exponents near zero would imply the absence of the continuous zero crossings we will need to satisfy conjecture (2). To satisfy conjecture (1) we only need three exponents to be near zero and undergo zero crossing for the minimal bifurcation chain subset\[75\] to exist. There are clearly enough exponents on average for such to exist for at least some interval in parameter space at \( d = 32 \), e.g. for \((0.1, 0.5)\). For \( d = 64 \) that interval is much longer — \((0.1, 1)\). Finally, if we want the chain link set to be more connected and for the distance between elements of the bifurcation chain subset to decrease, we will need the fraction of exponents near zero for the fixed interval \((-0.01, 0.01)\) for a given interval in \( s \) to increase with dimension. This figure does not imply that there will exist zero-crossings, but it provides the necessary circumstance for our arguments.

The second argument falls out of the \( a \)–density and \textit{num}–continuity arguments. We know that as the dimension is increased, the variation of Lyapunov exponents
versus parameter variation decreases until, at dimension 64, the exponent variation varies continuously within numerical errors (and thus upon moving through zero, the exponent moves through zero continuously). We also know that on the interval in parameter space \( A = [0.1, 6] \), the distance between exponent zero crossings decreases monotonically. Further, on this subset \( A \), there always exists a positive Lyapunov exponent, thus implying the existence of bifurcation chain set whose length is at least 5.9. Extrapolating these results to their limits in infinite dimensions, the number of exponent crossings on the interval \( A \) will monotonically increase with dimension. As can be seen from Fig. (14), the exponent zero-crossings are relatively uniform with the distance between crossings decreasing with increasing dimension. Considering Fig. (12), the exponent zero crossings are also transverse to the \( s \) axis. Thus the zero crossings on the interval \( A \), which are exactly the points of non-hyperbolicity we are searching for, are becoming dense. This is overkill for the verification of the existence of a minimal bifurcation chain set. This is strong evidence for both conjectures (1) and (2). It is worth noting that hitting these points of hyperbolicity violation upon parameter variation is extremely unlikely under any uniform measure on \( R \) as they are a countable collection of points.[76] Luckily, this does not matter for either the conjecture at hand or for any of our other arguments.

3. Relevance

The above argument provides direct numerical evidence of hyperbolicity violation over a range of the parameter space. This is strong evidence supporting conjecture (1). It does not yet verify conjecture (2), but it sets the stage as we have shown that there is a significant range over which hyperbolicity is violated. The former statement speaks to conjecture (4) also; a full explanation of conjecture (4) requires further analysis, which is the subject of a discussion in the final remarks.

C. Hyperbolicity violation versus parameter variation

We are finally in a position to consider the final arguments for conjecture (2). To complete this analysis, we will need the following pieces of information:

i. we need the maximum number of positive exponents to go to infinity

ii. we need a region of parameter space for which \( a \)-density of Lyapunov exponent zero crossings exists; i.e. we need an arbitrarily large number of adjoining bifurcation link sets (such that the cardinality of the bifurcation chain set becomes arbitrarily high) such that for each \( V_i \), the length of \( V_i \), \( l = |b_i - a_i| \), approaches zero.

iii. we need \( \text{num} \)-continuity of exponents to increase as the dimension increases

iv. a major simplification can be provided with the existence of one global maximum in the number of positive exponents and entropy, and along any portion of parameter space where \( s \) is greater than the \( s \) at the maximum number of positive exponents, the maximum and minimum number of exponents occur on the graph at the end points of the parameter range (within numerical accuracy)

The \( a \)-density, \( \text{num} \)-continuity and the arbitrary numbers of positive exponent arguments we need have, for the most part, been provided in previous sections. In this section we will simply apply the \( a \)-density and \( \text{num} \)-continuity results in a manner that suits our needs. The evidence for the existence of a single maximum in the number of positive exponents, a mere convenience for our presentation, is evident from section (V C). We will simply rely on all our previous figures and the empirical observation that as the dimension is increased above \( d = 32 \), for networks that have the typical \( \text{num} \)-continuity (which includes all networks observed for \( d \geq 64 \)), there exists a single, global maximum in the number of positive exponents versus parameter variation.

1. Qualitative analysis

The qualitative picture we are using for intuition is that of Fig. (12). This figure displays all the information we wish to quantify for many networks; as the dimension is increased, there is a region of parameter space where the parameter variation needed to achieve a topologically different (by topologically different, we mean a different number of global stable and unstable manifolds) attractor decreases to zero. Based on Fig. (12) (and hundreds of similar plots), we claim that qualitatively this parameter range exists for at least \( 0.5 \leq s \leq 6 \).

2. Quantitative and numerical analysis

Let us now complete our arguments for conjecture (2). For this we need a subset of the parameter space, \( B \subset R^3 \), such that some variation of \( s \in B \) will lead to a topological change in the map \( f \) in the form of a change in the number of global stable and unstable manifolds. Specifically, we need \( B = \bigcup \mathcal{V}_i = V \), where \( V_i \) and \( V_{i+1} \) share a limit point and are disjoint. Further, we need the variation in \( s \) needed for the topological change to decrease monotonically with dimension on \( V \). More precisely, on the bifurcation chain set, \( U \), the distance between elements must decrease monotonically with increasing dimension. We will argue in three steps: first, we will argue that, for each \( f \) with a sufficiently high number of dimensions, there will exist an arbitrarily large number of exponent zero crossings (equivalent to an arbitrarily
large number of positive exponents); next we will argue that the zero crossings are relatively smooth; and finally, we will argue that the zero crossings form an \( a \)-dense set on \( V \) — or on the bifurcation chain set, \( l = |b_i - a_j| \to 0 \) as \( d \to \infty \). This provides strong evidence supporting conjecture (2).

Assume a sufficiently large number of dimensions, verification of conjecture (1) gives us the existence of the bifurcation chain set and the existence of the adjoining bifurcation link sets. The existence of an arbitrary number of positive Lyapunov, and thus an arbitrarily large number of zero crossings follows from section (V C). That the bifurcation chain set has an arbitrarily large number of elements, \# \( U \to \infty \) is established by conjecture (3), because, without periodic windows, every bifurcation link set will share a limit point with another bifurcation link set. From section (V A), the \( \text{num} \)-continuity of the exponents persists for a sufficiently large number of dimensions, thus the Lyapunov exponents will cross through zero. Finally, section (V B) tells us that the Lyapunov exponent zero crossings are \( a \)-dense, thus, for all \( c_i \in U \), \( |c_i - c_{i+1}| \to 0 \), where \( c_i \) and \( c_{i+1} \) are sequential elements of \( U \).

Specifically for our work, we can identify \( U \) such that \( U \subset [0.5, 6] \). We could easily extend the upper bound to much greater than 6 for large dimensions (\( d \geq 128 \)). How high the upper bound can be extended will be a discussion in further work.

Finally, it is useful to note that the bifurcation link sets are LCE decreasing with increasing \( s \). This is not necessary to our arguments, but it is a nice added structure that aids our intuition. The LCE decreasing property exists due to the existence of the single, global maximum in the maximum number of positive Lyapunov exponents followed by an apparent exponential fall off in the number of positive Lyapunov exponents.

3. Relevance

The above arguments provide direct evidence of conjectures (2) and (4) for a one-dimensional curve (specifically an interval) in parameter space for our networks. This evidence also gives a hint with respect to the robustness of chaos in high-dimensional networks with perturbations on higher-dimensional surfaces in parameter space. Finally, despite the seemingly inevitable topological change upon minor parameter variation, the topological change is quite benign.

VII. FITTING EVERYTHING TOGETHER

Having finished with our specific analysis, it is now time to put our work in the context of other work, both of a more mathematical and a more practical and experimental nature. In this spirit, we will provide, first, a brief summary of our arguments followed by a discussion of how our results fit together with various theoretical results from dynamical systems and turbulence.

A. Summary of arguments

We will give brief summaries of our results, both in the interest of clarity and to relate our results and methods to others.

1. Periodic window probability decreasing: conjecture 3

The conjecture that the probability of periodic window existence for a given \( s \) value along an interval in parameter space decreases with increasing dimension upon the smallest positive Lyapunov exponent becoming positive, is initially clear from considering the Lyapunov spectra of neural networks versus parameter variation for networks of increasing size (Figs. (7) and (8)). We show that as the dimension is increased, the observed probability of periodic windows decreases inversely with increase in dimension. The motivation for arguing in this way is simple; this analysis is independent from the \( \text{num} \)-continuity analysis, and the results from the analysis of \( \text{num} \)-continuity and periodic window probability decrease reinforce each other. The mechanism that this conjecture provides us with is the lengthening of the bifurcation chain set.

Further investigations of this particular phenomena will follow in a later report. For other related results see [52], [57], [58], and [56].

2. Hyperbolicity violation: conjecture 1

The intuition for this conjecture arises from observing that for our high-dimensional systems, there exists at least one Lyapunov exponent that starts negative, becomes positive, then goes negative again; thus if it behaves numerically continuously, it must pass through zero for some parameter value \( s \).

To verify this conjecture, we presented two different arguments. This first argument was a necessary but not sufficient condition for hyperbolicity violations. We show that over a sizeable interval in parameter space, there exists a Lyapunov exponent very near zero, and the fraction of the total number of Lyapunov exponents that are near zero increases over a larger interval of parameter space as the dimension is increased. The second argument was based on the \( a \)-density of exponent zero crossings, the \( \text{num} \)-continuity of the exponents as the dimension increased, and the increasing number of positive exponents with dimension. Both arguments together help imply an interval of parameter space such that on that interval, the number of parameter values such that hyperbolicity is violated is increasing.
3. **Existence of Codimension-$\epsilon$ bifurcation set: conjecture 2**

Conjecture (2) is the next step in relating our results with the results of structural stability theory. Given the results supporting conjecture (1), conjecture (2) only needs a few added bits of evidence for its vindication.

The intuition for this argument follows from observing that the peak in the number of positive Lyapunov exponents tends toward a spike of increasing height and decreasing width as the dimension is increased. This, with some sort of continuity of exponents, argues for a decrease in distance between exponent zero crossings.

A summary for the arguments regarding conjecture (2) is as follows. With increasing dimension we have: increased num-continuity of Lyapunov exponents; increasing number of positive Lyapunov exponents; and a-density of Lyapunov exponent zero crossings (thus all the exponents are not clustered on top of each other). Thus, on a finite set in parameter space, we have an arbitrary number of exponents that move numerically smoothly from negative values, to positive values, and back to negative values. Further, these exponents are relatively evenly spaced. Thus, the set in parameter space for which hyperbolicity is violated is increasingly dense; and with an arbitrarily number of violations available, the perturbation of the parameter required to force a topological change (a change in the number of positive exponents) becomes small.

4. **Non-genericity of structural stability: conjecture 4**

As previously mentioned, it could appear that our results are contrary to Robbin [2], Robinson [3], and Mañé [4]. We will discuss specifically how our results fit with theirs in section (VII C 1). In the current discussion, we wish to properly interpret our results in a numerical context.

We claim to have found a subset of parameter space that, in the limit of infinite dimensions, has dense hyperbolicity violation. This could be interpreted to imply that we have located a set for which strict hyperbolicity does not imply structural stability, because the $C^1$ changes in the parameter give rise to topologically different behaviors. The key issue to realize is that in numerical simulations, there do not exist infinitesimals or infinite-dimensional limits[77]. Rather, we can speak to how behaviors arise, and how limits behave along the path to the ideal. We have found a subset of parameter space that we believe can approximate (with unlimited computing) arbitrarily closely a set for which hyperbolicity will not imply structural stability. Thus, an experimentalist or a numerical physicist might see behavior that looks like it violates the results of Robbin [2], Robinson [3], and Mañé [4]; yet it will not strictly be violating those theorems. The key point of this conjecture is that we can observe apparent violation of the structural stability conjecture, but the violation (on a Lebesgue measure zero set) occurs as smooth, not catastrophic, topological change. (In section (VII C 1) we will further discuss our results as they relate to those of Robbin [2], Robinson [3], and Mañé [4].)

5. **Robust chaos: conjecture 5**

That chaos is a robust behavior for bounded high-dimensional dynamical systems is not particularly surprising, especially in light of Fig. (4), information presented in sections (V C) and (VI A), and previous work [24]. Beyond casual observation, we will not comment because it is the topic of a work in progress [56]. It is important to note that we do not observe sinks or periodic windows in the chaotic region of parameter space for a sufficiently high dimension. This particular characteristic is, however, somewhat heartening if one is to compare our results with many high-dimensional chaotic and turbulent natural systems as these systems are constantly being perturbed, yet their behavior is relatively robust. Readers interested in arguments along the lines of conjecture (5) are directed to [52], [57], [58], [56], or [59] for further information.

B. **Fitting our results in with the space of $C^\alpha$ function: how our network selection method affects our view of function space**

Performing a numerical experiment induces a measure upon whatever is being experimented upon. We now discuss some of the characteristics of our imposed measure and how they might affect our results. Recall, often in mathematics, it is desirable to prove that various results are invariant to the measure imposed upon the space; in our case this would be extremely difficult if not impossible, thus we will resort to the aforementioned, standard experimental style.

A measure, in a very general sense, provides a method of measuring the volume a set occupies in its ambient space (for a formal treatment, see [60]). Usually that method provides a specific mechanism of measuring lengths of a covering interval. Then, the entire space is covered with the aforementioned intervals, and their collective volume is summed. One of the key issues is how the intervals are weighted. For instance, considering the real line with the standard Gaussian measure imposed upon it; the interval $[-1, 1]$ contains the majority of the volume of the entire interval $[-\infty, \infty]$. Our method of weighting networks selects fully connected networks with random Gaussian weights. Thus, in limit of high dimension and high number of neurons, very weakly connected networks will be rare, as the Gaussian statistics of the weights will be dominant. Likewise, fully connected networks where all the weights have the same strength (up to an order of magnitude) will also be uncommon. One can argue whether our measure realistically represents
the function space of nature, but those arguments are ill-formed because they cannot be answered without either specific information about the natural system with which our framework is being compared, or the existence of some type of invariant measure. Nevertheless, our framework does cover the entire space of neural networks noted in section (I), although all sets do not have equal likelihood of being selected, and thus our results must be interpreted with this in mind.

A second key issue regards how the ambient space is split into intervals; or in a numerical sense, how the grain of the space is constructed. We will again introduce a simpler case for purposes of illustration, followed by a justification of why the simpler case and our network framework are essentially equivalent. Begin with $\mathbb{R}^n$ and select each coordinate $(v_i)$ in the vector $v = \{v_1, v_2, \ldots, v_n\} \in \mathbb{R}^n$ from a normal, i.i.d. distribution with mean zero, variance one. Next, suppose that we are attempting to see every number and every number combination. This will be partially achieved by the random number selection process mentioned above, and it is further explored by sweeping the variance, i.e., selecting a scalar $s \in \mathbb{R}$, $0 < s$, and sweeping $s$ over the positive real line, $sv$. This establishes two meshes, one for the individual vectors which is controlled by how finely the $s$ parameter is varied, and another mesh that controls how the initial coordinates are selected. These two combined meshes determine the set of combinations of coordinates that will be observed. If one considers how this affects vector selection in, say, $\mathbb{R}^3$ for simplicity, both in the initial vector selection and in the vector sweeping, it is clear how $\mathbb{R}^3$ will be carved out.

The point of the above paragraph is simply this: we are associating how we carve up our neural network function space with how we carve up the neural network weight space. It should be clear that this is comparing apples to apples. In the above paragraph, to understand how our neural network selection process works, simply associate $v$ with the vectors in the $w$ matrix and the scaling parameter $s$ with $s$. This keeps the view of our function space largely in standard Euclidean space. Of course there is the last remaining issue of the amplitude terms, the $\beta$'s. Apply the same type of analysis to the $\beta$'s as we did for the $w$'s in the above paragraph. Of course initially it would seem that the scaling parameter is missing, but note that multiplying the $\beta$'s by $s$, in our networks, is essentially equivalent to multiplying the $w$'s by $s$. To understand this, consider the one-dimensional network, with one neuron:

$$x_{t+1} = \beta_0 + \beta_1 \tanh(sw_0 + sw_1 (\beta_0 + \beta_1 \tanh(sw_0 + sw_1 x_{t-1})))$$

(39)

It is clear from this that inserting $s$ inside tanh will sweep the $\beta$'s, but inserting $s$ outside the squashing function will miss sweeping the $w_0$ bias term.\[78\]

From this is should be clear that our framework will capture the entire space of neural networks we are employing. Yet, it should also be clear that we will not select each network with equal probability. Weakly connected networks will not be particularly common in our study, especially as the number of dimensions and neurons increase, because the statistics of our weights will more closely resemble their theoretical distributions. It is also worth noting that a full connection between network structure and dynamics, in a sensible way, is yet out of reach (as opposed to, say, for spherical harmonics). Nevertheless, we claim that our framework gives a complete picture of the space of $C^r$ maps of compact sets to compact sets with the Sobolev metric from the perspective of a particular network selection method.

C. Our results related to other results in dynamical systems

As promised throughout, we will now connect our results with various theorems and conjectures in the field of dynamical systems. This will hopefully help put our work in context and increase its understandability. We will address how our work fits in with the stability conjecture of Smale and Palis [1]. First we will discuss our results and the structural stability theories of Robbin [2] and Mañé [4] which state that structurally stable systems must be hyperbolic. We will follow this by relating our studies to the work in partial hyperbolicity and stable ergodicity – the reaction to difficulties in showing that hyperbolic systems are structurally stable. We will conclude this portion of the summary by discussing how our work relates to one of the conjectures from a paper by Palis [7].

1. Structural stability theory and conjecture 4

It is now time to address the apparent conflict between our observations and the structural stability theorems of Robbin [2], Robinson [3], and Mañé [4]. We would like to begin by noting that we do not doubt the validity or correctness of any of the aforementioned results. In fact, any attempt to use our techniques and results to provide a counter example to the theorems of Robbin, Robinson, or Mañé involves a misunderstanding of what our methods are able to do and indeed intend to imply.

In conjecture (4) we claim, in an intuitive sense, that along a one-dimensional curve in parameter space, our dynamical systems are hyperbolic with measure one, with respect to Lebesgue measure. Yet, we can still find subsets that are measure zero, yet $a$–dense, for which our dynamical systems are partially hyperbolic rather than hyperbolic. The motivation for the above statement roughly derives from thinking of a turbulent fluid. In this circumstance, the number of unstable manifolds can be countably infinite, and upon varying, say, the viscosity, from very low to very high, one would have a countable number of exponents becoming positive over a finite length of parameter space. Yet, all the limits of this sort and all the intuitive ideas with respect to what will hap-
pen in the infinite-dimensional limit, are just that, ideas. There are limits to what we can compute; there do not exist infinite-dimensional limits in numerical computing; there do not exist infinitesimals in numerical computing; and aside from the existence of convergence theorems, we are left unable to draw conclusions beyond what our data says. Thus, our results do not provide any sort of counterexample to the stability conjecture. Rather, a key point of our results is that we do observe, in a realistic numerical setting, structural instability upon small parameter variation. It is useful to think instead of structural stability as an open condition on our parameter space whose endpoints correspond to the points of structural instability - the points of bifurcations in turbulence. These disjoint open sets are precisely the bifurcation link subsets, $V_i$ for which the map $f$ is structurally stable. As the dimension is increased, the length of the $V_i$’s decreases dramatically, and may fall below numerical or experimental resolution. Thus, the numerical or experimental scientist might observe, upon parameter variation, systems that should according to the work of Robbin, Robinson and Mañé, be structurally stable, to undergo topological variation in the form of a variation in the number of positive Lyapunov exponents; i.e. the scientist might observe structural instability. This is the very practical difference between numerical computing and the world of strict mathematics. (Recall we were going to attempt to connect structural stability theory closer to reality, the former statement is as far as we will go in this report.) The good news is that even though observed structural stability might be lost, it is lost in a very meek manner - the topological changes are very slight, just as seems to be observed in many turbulent experimental systems. Further, partial hyperbolicity is not lost, and the dynamically stable characteristics of stable ergodicity seem to be preserved, although we obviously can’t make a strict mathematical statement.

Thus, rather than claiming our results are contrary to those of Robbin [2], Robinson [3], and Mañé [4], we note that our results speak both to what might be seen of those theorems in high-dimensional dynamical systems and how their results are approached upon increasing the dimension of a dynamical system.

It is worth noting that, given a typical 64-dimensional network, if we fixed $s$ at such a point that there was an exponent zero crossing, we believe (based upon preliminary results) that there will exist many perturbations of other parameters that leave the exponent zero crossing unaffected. However, it is believed at this time that these perturbations are of very small measure (with respect to Lebesgue measure), and of a small codimension set, in parameter space, i.e. we believe we can find perturbations that will leave the seemingly transversal intersection of an exponent with 0 at a particular $s$ value unchanged, yet these parameter changes must be small.

2. Partial hyperbolicity

In this study we are particularly concerned with the interplay, along a parameterized curve, of how often partial hyperbolicity is encountered versus strict hyperbolicity. It should be noted that if a dynamical system is hyperbolic, it is partially hyperbolic. All of the neural networks we considered were at least partially hyperbolic; we found no exceptions. Many of the important questions regarding partially hyperbolic dynamical systems lies in showing the conditions under which such systems are stably ergodic. We will now discuss this in relation to our results and methods.

Pugh and Shub [28] put forth the following conjecture regarding partial hyperbolicity and stable ergodicity:

**Conjecture 6 (Pugh and Shub [28] Conjecture 3)**
Let $f \in \text{Diff}^{s}_{\mu}(M)$ where $M$ is compact. If $f$ is partially hyperbolic and essentially accessible, then $f$ is ergodic.

In that same paper they also proved the strongest result that had been shown to date regarding their conjecture:

**Theorem 3 (Pugh-Shub theorem (theorem A [28]))** If $f \in \text{Diff}^{s}_{\mu}(M)$ is a center bunched, partially hyperbolic, dynamically coherent diffeomorphism with the essential accessibility property, then $f$ is ergodic.

A diffeomorphism is partially hyperbolic if it satisfies the conditions of definition (7). Ergodic behavior implies that, upon breaking the attractor into measurable sets, $A_f$, for $f$ applied to each measurable set for enough time, $f^n(A_f)$ will intersect every other measurable set $A_j$. This implies a weak sense of recurrence; for instance, quasi-periodic orbits, chaotic orbits, and some random processes are at least colloquially ergodic. More formally, a dynamical system is ergodic if and only if almost every point of each set visits every set with positive measure. The accessibility property simply formalizes a notion of one point being able to reach another point. Given a partially hyperbolic dynamical system, $f : X \to X$ such that there is a splitting on the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, and $x, y \in X$, $y$ is accessible from $x$ if there is a $C^1$ path from $x$ to $y$ whose tangent vector lies in $E^s \cap E^u$ and vanishes finitely many times. The diffeomorphism $f$ is center bunched if the spectra of $Tf$ (as defined in section (II B)) corresponding to the stable ($T^s f$), unstable ($T^u f$), and ($T^c f$) central directions lie in thin, well separated annuli (see [15], page 131 for more detail, the radii of the annuli is technical and is determined by the Holder continuity of the diffeomorphism.) Lastly, let us note that a dynamical system is called stably ergodic if, given $f \in \text{Diff}^{s}_{\mu}(M)$ (again $M$ compact), there is a neighborhood, $f \in Y \subset \text{Diff}^{s}_{\mu}(M)$ such that every $g \in Y$ is ergodic with respect to $\mu$. We will refrain from divulging an explanation of dynamical coherence; it is a very crucial characteristic for the proof of theorem (3), but we will have little to say in its regard.

An actual numerical verification of ergodicity can be somewhat difficult as the modeler would have to watch
each point and verify that eventually the trajectory re
turned very close to every other point on the orbit (i.e. it satisfies the Birkhoff hypothesis). Doing this for a few points is, of course, possible, but doing it for a high-
dimensional attractor for any sizable number of points can be extremely time consuming. Checking the acces-
sibility criterion seems to pose similar problems - in fact it is hoped that accessibility is the sufficient recurrence conditions for ergodic behavior - thus is should be no surprise that accessibility would be difficult to check nu-
ermically (it has been shown to be $C^1$ dense [61]). In the reality of computing, there is a far more practical way of checking for ergodic behavior, motivated by a more prac-
tical problem in numerical computing, transients. For a mathematician, ergodic tools can be applied whenever the system can be shown to be ergodic. In numerical work, proving that the necessary conditions for the use of ergodic measures is often intractable. Besides, for nu-
umerical applications, proving long-term behavior is often not good enough since the use of an ergodic diagnostic, for the relaxation from the transients to the ergodic state can, at times, be prohibitively slow, and sometimes diffi-
cult to detect. There are even times when the numerical errors in the calculations effectively reset the transients. The practical solution to this is to apply the ergodic mea-
sures and, along with the time-series data, watch the transients disappear. We did this specifically in section (IV) to justify our use of ergodic measures. If the errors in the ergodic measures along with the transients of the attractors decrease with time, then we call the system ergodic and feel justified in using ergodic measures, such as Lyapunov exponents.

Considering Figs. (4), (5), and (6), it seems clear that our networks are ergodic since the ergodic measures con-
verge. Further, upon considering Figs. (7), (8), and (12), when a one-dimensional parameter is varied, er-
godic behavior is preserved. Of course, showing that one has explored all the variations inside the neighborhood $(f) \in Y \subset \text{Diff}^2(M)$ is impossible: thus claiming that we have, in a mathematically rigorous way, observed sta-
ble ergodicity as the predominant characteristic would be premature. Further, we can say little about the accessi-
ability property. What we can say is that we have never observed a dynamical system, within our construction, that is not on a compact set, is not partially hyperbolic, and is not stably ergodic. Thus, our results provide evi-
dence that the conjecture of Shub and Pugh is on track. For more information with respect to the mathematics discussed above, see [15], [5], or [27].

Comparing conjecture (6) to theorem (3), the required extra hypotheses for the proof of the theorem are dy-
namical coherence and center bunching of the spectrum of $T_f$. Pugh and Shub, and others have been attempting to eliminate these extra hypothesis. Our results speak lit-
tle to the issue of dynamic coherence, but our results can speak to the issue of center bunching. Considering Fig. (8) at any value of the $s$ parameter, there is no evidence of center bunching, or any real bunching of Lyapunov ex-
ponents at all. In fact, if there were center bunching, our $a$–density of exponent zero crossing argument would be in serious trouble. Thus, we claim that we have strong evidence for the removal of the center bunching require-
ment for stable ergodicity. And, since we are claiming that our dynamical systems are seemingly ergodic, if cen-
ter bunching were required of stable ergodicity, we claim that stable ergodicity would be too strict of a distinguishes-
characteristic for dynamic stability.[79]

3. Our results and Palis’s conjectures

Palis [7] stated many stability conjectures based upon the last thirty years of developments in dynamical sys-
tems. We wish to address one of his conjectures:

**Conjecture 7 (Palis [7] conjecture II)** In any di-

**dimension, the diffeomorphisms exhibiting either a homo-
clinic tangency or a (finite) cycle of hyperbolic periodic

**orbits with different stable dimensions (heterodimensional cycle) are $C^r$ dense in the complement of the closure of the hyperbolic ones.**

Let us decompress this for a moment, and then dis-
cuss how our results fit with it. Begin by defining the space of $d$-dimensional $C^r$ diffeomorphisms as $X$. Next, break that space up as follows: $A = \{x \in X|x$ exhibits a homoclinic tangency or a finite cycle of hyperbolic periodic orbits with different stable dimensions $\}$ and $B = \{x \in X|x$ is hyperbolic $\}$. Thus $B$ is the set of hyper-

**bolic, aperiodic diffeomorphisms, and $A$ is the set of 

**periodic orbits or partially hyperbolic orbits. The con-

**jecture states that $A$ is dense in the complement of the closure of $B$; thus $A$ can be dense in $B$. With respect to our results, the partially hyperbolic diffeomorphisms (dif-

**feomorphisms with homoclinic tangencies) can be dense 

**within the set of hyperbolic diffeomorphisms. Our con-

**jectures claim to find a subset of our one-dimensional pa-

**rameter space such that partially hyperbolic diffeomor-

**phisms will, in the limit of high dimensions, be dense. In 

**other words, our work not only agrees with Palis’s con-

**jecture II (and subsequently his conjecture III), but our 

**work provides evidence confirming Palis’s conjectures. Of 

**course, we do not claim to provide mathematical proofs, 

**but rather strong numerical evidence supporting Palis’s ideas.

D. Final remarks

Finally, let us briefly summarize:

**Statement of Results 2 ((Summary)) Assuming 

**our particular conditions and our particular space of $C^r$ dynamical systems as per section I, there exists a collection of bifurcation link subsets ($V$) such that, in the limit of countably infinite dimensions, we have numerical evidence for the following:
Conjecture 1: on the above mentioned set $V$, strict hyperbolicity will be violated $a \rightarrow \text{densely}$.

Conjecture 2: on the above mentioned set $V$, the number of stable and/or unstable manifolds will change under parameter variation below numerical precision.

Conjecture 3: on the above mentioned set $V$, the probability of the existence of a periodic window for a given parameter interval decreases inversely with dimension.

Conjecture 4: on the above mentioned set $V$, hyperbolic dynamical systems are not structurally stable within numerical precision with measure one with respect to Lebesgue measure in parameter space.

In a measure-theoretic sense hyperbolic systems occupy all the space, but the partially hyperbolic dynamical systems (with non-empty center manifolds) can be $a \rightarrow \text{dense}$ on $V$. Intuitively, if there are countable dimensions - thus countable Lyapunov exponents, then one of two things can happen upon parameter variation:

i. there would have to be a persistent homoclinic tangency- or some other sort of non-transversal intersection between stable and unstable manifolds that was persistent to parameter changes;

ii. there can be, at most, countable parameter points such that there are non-transversal intersections between stable and unstable manifolds.

We also see that for our networks, each exponent in the spectrum converges to a unique (within numerical resolution) value. This both confirms the usefulness and validity of our techniques, and provides strong evidence for the prevalence of ergodic behavior. Further, upon parameter variation, the ergodic behavior is seemingly preserved; thus we also have strong evidence of a prevalence of stable ergodic behavior.

VIII. ACKNOWLEDGMENTS

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In a practical sense, the

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\( \Omega f = \{ x \in M \forall \text{neighborhood } U \text{ of } x, \exists n \geq 0 \text{ such that } f^n(U) \cap U \neq \emptyset \} \)

\( E^0 \) is trivial, \( f \) is simply Anosov, or strictly hyperbolic.

In a practical sense, the \( x \) variation is the initial separation or perturbation of \( x \).

Note, there is an important difference between the Lyapunov exponent contracting, which implies some sort of convergence to a particular value, versus a negative Lyapunov exponent that implies a contracting direction on the manifold or in phase space.

When an exponent is very nearly zero it can tend to fluctuate above and below zero, but it is always very near zero. Thus although it might be difficult to resolve zero exactly — which is to be expected — the exponent is clearly very near zero which is all that really matters for our purposes.

The mean \( \nu \text{—continuity for } d = 4 \text{ and } d = 128 \text{ is } 0.015 \pm 0.03 \text{ and } 0.004 \pm 0.003 \text{, respectively. The mean } \nu \text{—continuity of the largest exponent for } d = 4 \text{ and } d = 128 \text{ is } 0.01 \pm 0.03 \text{ and } 0.002 \pm 0.004 \text{, respectively. The discrepancy between these two data points comes from the large error in the negative exponents at high dimension.}

Recall, the bifurcation chain sets will not exist when the zero crossings are not transverse.

We will save region \( I \) for a different report. For insight into some of the dynamics and phenomena of region \( I \), see [54].

This increase in smoothness is not necessarily a function of an increased number of exponents. A dynamical system that undergoes massive topological changes upon parameter variation will not have a smooth curve such as in Fig. (13), regardless of the number of exponents.

Further evidence for such an increase is provided by considering the Kaplan-Yorke dimension versus \( d \). Such analysis yields a linear dependence, \( D_{K-\nu} \propto d/2 \).

By neutral direction we mean a zero Lyapunov exponent; we do not wish to imply that there will always exist a center manifold corresponding to the zero Lyapunov exponent.

It is possible that there exist Milnor style attractors for our high-dimensional networks, or at least multiple basins of attraction. As this issue seems to not contribute, we will save this discussion for a different report.

The minimal bifurcation chain subset requires at least two adjoining bifurcation link sets to exist.

When considering parameter values greater than the point where the smallest exponent that becomes positive, becomes positive, the zero crossings seem always to be transverse. For smaller parameter values - along the route to chaos, a much more complicated scenario ensues.

Note that in previous sections we do use words like, “in the infinite-dimensional limit.” It is for reasons such as these (and many others) that we are only putting forth conjectures and not theorems; this distinction is not trivial.

Recalling section (IIA 4), we actually do a little more with the \( \beta \)'s than we mention here; the previous argument is simply meant to give a (mathematically imprecise) picture of how our experiment carves out the space of neural networks.

It is worth noting that in section 31 of Landau and Lifshitz’s fluid mechanics book [62], they give physical reasons why one might expect a center bunching type of spectrum, or at least a finite number of exponents near zero, in turbulent fluids.