# THEORY AND SIMULATION OF CYCLOTRON HEATING IN A LINEAR OCTUPOLE

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## ABSTRACT

A simple conductivity model is used to calculate the cyclotron heating rate (electron or ion) in an arbitrary non-uniform magnetic field. The result is applied to an ideal linear octupole field, which, because of its complexity, offers a severe test of the theory. The theory is compared with the results of a numerical simulation in which the trajectories and average energy of a collection of non-interacting particles are calculated as a function of time. The theory and simulation are in surprisingly good agreement over a wide range of heating frequency, plasma temperature, electric field strength, and wave number.

## Introduction

Many approaches have been used to calculate the cyclotron heating rate for plasmas in various magnetic field configurations. This paper presents a method of calculating cyclotron heating rates that is based on integration of the local plasma conductivity over a volume in which the magnetic field may vary arbitrarily. The result can readily be applied to any magnetic field configuration.

To test the validity of the theory over a wide range of parameters, a computer code was written to calculate the trajectories and average kinetic energy of a collection of non-interacting particles in an ideal linear octupole magnetic field. The octupole field was chosen because of its similarity to experimental devices at Wisconsin, and because its complexity insures a severe test of the theory.

## Theory

The power absorbed per unit volume by a plasma can be written in terms of the tensor conductivity $\overleftrightarrow{\sigma}$ as

$$\frac{dP}{dV} = (\overleftrightarrow{\sigma} \cdot \vec{E}) \cdot \vec{E} = \sigma_{\perp} E_{\perp}^2 + \sigma_{\parallel} E_{\parallel}^2 \ .$$

The spatially averaged particle heating rate is obtained by integrating over the volume and dividing by the total number of particles:

$$\frac{d\overline{W}}{dt} = \frac{\int \sigma E^2 dV}{\int n dV}$$

For a cold plasma in a uniform magnetic field the perpendicular and parallel components of the conductivity are

$$\sigma_{\perp} = \varepsilon_o \omega_p{}^2 \nu \left[\frac{\omega^2 + \omega_c{}^2 + \nu^2}{(\omega^2 - \omega_c{}^2)^2 + \nu^2(2\omega^2 + 2\omega_c{}^2 + \nu^2)}\right]$$

and

$$\sigma_{\parallel} = \frac{\varepsilon_o \omega_p{}^2 \nu}{\omega^2 + \nu^2} ,$$

where $\omega$ is the rf frequency, $\omega_c$ is the cyclotron frequency, $\omega_p$ is the plasma frequency, and $\nu$ is the collision frequency.

For $\nu \ll \omega$ it is clear that most of the heating occurs very near the resonance at the cyclotron frequency, and in fact, the heating rate can be expressed in terms of a $\delta$-function as

$$\frac{d\overline{W}}{dt} = \frac{\pi e}{2} \frac{\int n E_{\perp}{}^2 \ \delta(B-B_o) dV}{\int n dV} ,$$

where $B_o$ is the magnetic field at cyclotron resonance.

For many purposes it is useful to define a dimensionless heating rate $G$ according to

$$\frac{d\overline{W}}{dt} = \frac{e\overline{E^2}}{B_o} \ G ,$$

so that $G$ is given by

$$G = \frac{\pi B_o \int n (E_{\perp}{}^2/\overline{E^2}) \delta(B-B_o) dV}{2 \int n dV} .$$

For the special case in which the density is constant and the electric field is homogeneous and isotropic, G has a simple physical interpretation as the ratio of resonance volume to total volume:

$$G = \frac{\pi}{3} \frac{B_o}{V} \left.\frac{dV}{dB}\right|_{B_o} .$$

Note that the calculation applies equally well to electrons or ions provided the appropriate $B_o$ is used. Note also that the heating rate is independent of collision frequency in the limit $\nu \ll \omega$. The validity of this result was tested by calculating numerically the value of $G$ for

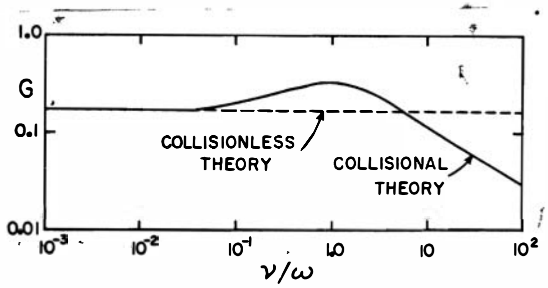


Figure 1

arbitrary $\nu$ for the special case of resonance near the axis of a linear octupole where $B(r) = (r/a)^3 B(a)$. Figure 1 shows a slight enhancement of the heating above $\nu/\omega \sim 0.1$ followed by a reduction in heating at very large collision rates.

## Simulation

Since the theoretical model is based on a conductivity derived for a zero temperature plasma in a uniform magnetic and electric field, we expect limits on the range of validity of the theory. To test these limits, a computer code was written to solve the exact 3-D equations of motion of 100 non-interacting, non-relativistic particles (either electrons or ions) in a linear octupole field such as shown in Figure 2. The field is produced by four, infinitely long, infinitessimal current filaments located at ($\pm a$, $\pm a$) in a cartesian coordinate system centered on the axis. Such a field offers a severe test of the theory since the field strength varies from zero on the axis to infinity at the conductors. The particles are started at random positions over the square cross section with a constant but randomly oriented velocity vector. The rf electric field is uniform and in the $z$ (ignorable) direction, except where otherwise noted.

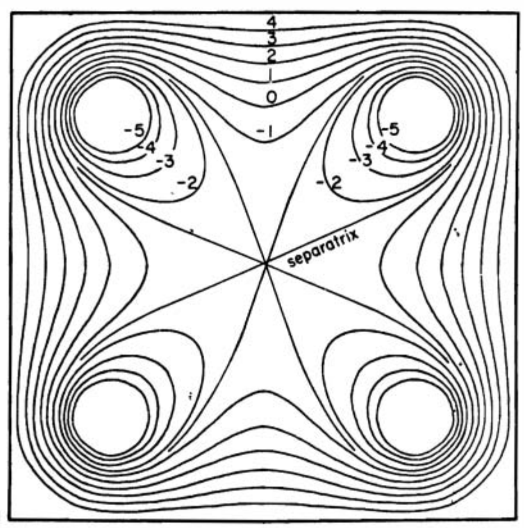


Figure 2

Figure 3 shows the average energy of the 100 particles as a function of time along with the theoretical prediction for a typical case with $E/c\,B_o = .01$ and $B_o = B(a)$. In the subsequent figures, the heating rate is determined from the simulation by the slope of a straight line that is a least square fit to the data points.

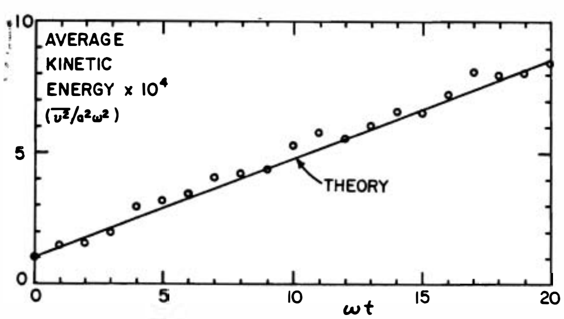


Figure 3

Figure 4 shows the result of varying the position of the resonance zone. For small $B_o$ (resonance near the axis) and large $B_o$ (resonance near the conductors) an asymptotic form of the theory can be derived analytically. For intermediate $B_o$ (resonance near the walls), the theory must be evaluated numerically. The circles show that the heating observed in the simulation generally follows the prediction.

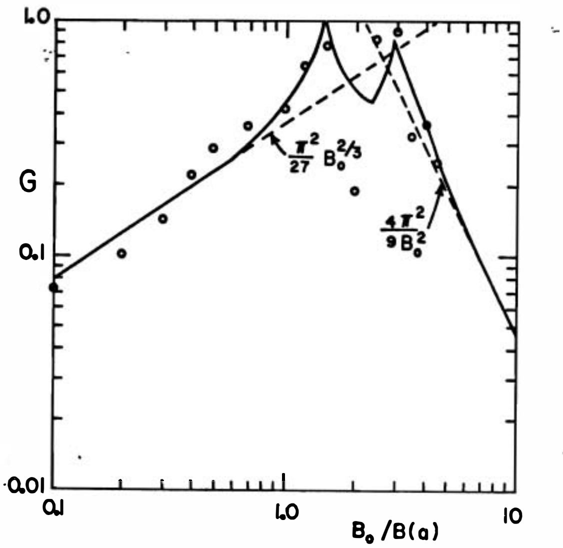


Figure 4

Figure 5 shows the result of changing the initial energy of the particles (expressed in terms of the gyroradius at resonance, $\rho_o$). In this case the electric field strength was also changed so that the relative growth in energy during the simulation is about constant. The results agree remarkably well with the theory until the gyroradius becomes comparable to the size of the machine.

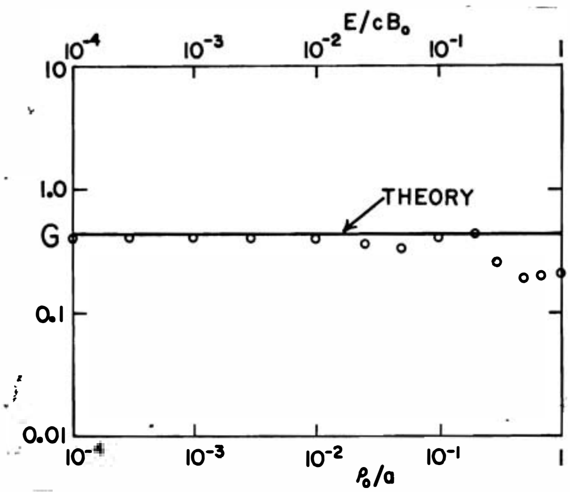


Figure 5

In Figure 6 the electric field was oriented in the y (vertical) direction and given a cos kx dependence to simulate a cavity mode. The rf magnetic field produced by this electric field was also included. Note that in this case, there are also components of $\tilde{E}$ parallel to $\tilde{B}$. Perhaps the most remarkable result of this paper is that within about a factor 2 the theory correctly predicts the heating observed in the simulation even when the gyroradius becomes much larger than the wavelength. This result may be fortuitous, however.

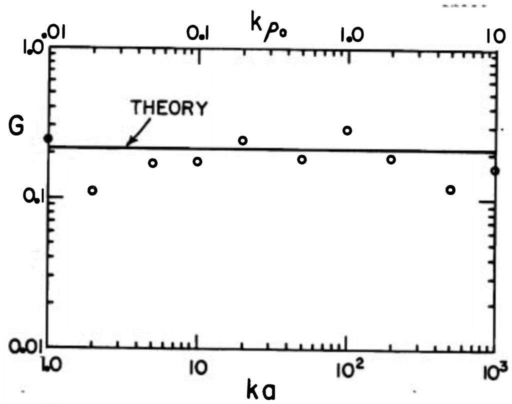


Figure 6

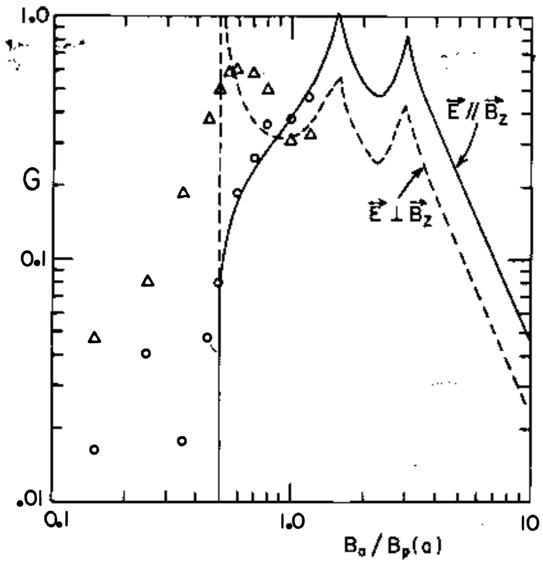


Figure 7

In Figure 7 a uniform $\vec{B}$ field was added in the z-direction ($B_z = 0.5\ B_p(a)$) to approximate the toroidal field in toroidal experiments. In this case there is a minimum $B_o$ below which no heating should occur. The theory predicts a singularity in heating at $B_o = B_z$ if $\vec{E}$ is perpendicular to $B_z$. The simulation results show general agreement except that the singularity is noticeably damped (Δ's), and both cases exhibit some heating below resonance. Even this result can be explained, at least qualitatively, by postulating an effective collision frequency ν that is a few tenths of ω.


Summary

A theoretical model based on integration of the conductivity tensor over the volume of a plasma gives an expression for the cyclotron heating rate that can be applied to any magnetic field configuration. The theory has been tested by comparing with the results of a computer simulation of single particle trajectories in a linear octupole magnetic field. The results are in remarkably good agreement over a wide range of rf frequency, wavelength, and particle energy.


Acknowledgment


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