Interpretations of Electron Energy Distribution

by

J. C. Sprott

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Kuswa (PLP 286 and 314) has observed that the energy distribution function for electrons on the B = 0 axis in the toroidal octupole during microwave heating of a gun plasma at low background gas pressure is relatively constant in time and has the form

$$f(W) \sim e^{-\sqrt{W}}$$

for a variety of heating conditions. This observation suggests a balance between the heating and loss rates of electrons of a given energy. We will consider here various combinations of heating and loss mechanisms and calculate the resulting equilibrium distribution functions.

Consider an isotropic distribution $f(W, \psi, t)$, such that

$$\frac{df}{dt} = \frac{\partial f}{\partial W}\frac{dW}{dt} + \frac{\partial f}{\partial \psi}\frac{d\psi}{dt} + \frac{\partial f}{\partial t} .$$

If we neglect diffusion $(\frac{d\psi}{dt} = 0)$, and require an equilibrium solution $(\frac{\partial f}{\partial t} = 0)$, $f(W)$ is given by

$$f(W) \sim e^{-\int \frac{\gamma(W)\,dW}{dW/dt}} ,$$

where $\gamma(W)$ is the loss rate of electrons of energy W:

$$\frac{df}{dt} = -\gamma f.$$

For $\gamma \sim W^{\ell}$ and $\frac{dW}{dt} \sim W^{h}$, we obtain

$$f(W) \sim \begin{cases} e^{W^{1 + \ell - h}} & \text{for } h - \ell > 1 \\ W^{-c} & \text{for } h - 1 = 1. \end{cases}$$

Solutions with $h - \ell < 1$ are unphysical and cannot produce an equilibrium. The distribution functions calculated for various combinations of heating and loss mechanisms are given in the following table:

TABLE

	$\frac{dW}{dt}$ →	W^{-1}	$W^{-\frac{1}{2}}$	const	$\sqrt{W}$	W
γ ↓				Stochastic	Resonant	
$W^{-3/2}$	Coulomb Scatt	$e^{-\sqrt{W}}$	W^{-c}			
W^{-1}		e^{-W}	$e^{-\sqrt{W}}$	W^{-c}		
$W^{-\frac{1}{2}}$		$e^{-W^{3/2}}$	e^{-W}	$e^{-\sqrt{W}}$	W^{-c}	
const	Convection	e^{-W^2}	$e^{-W^{3/2}}$	e^{-W}	$e^{-\sqrt{W}}$	W^{-c}
$\sqrt{W}$	Obstacles	$e^{-W^{5/2}}$	e^{-W^2}	$e^{-W^{3/2}}$	e^{-W}	$e^{-\sqrt{W}}$
W	"Bohm"	e^{-W^3}	$e^{-W^{5/2}}$	e^{-W^2}	$e^{-W^{3/2}}$	e^{-W}

The heating is stochastic if the electrons transiently and periodically pass through resonance, and resonant if the electrons remain trapped in a region of resonance at a local magnetic well. Stochastic heating gives $f(W) \sim e^{-\sqrt{W}}$ for $\gamma \sim W^{-\frac{1}{2}}$, and resonant heating gives $f(W) \sim e^{-\sqrt{W}}$ for $\gamma = \text{const}$. Kuswa (PLP 286 and 314) has observed that when the heating is turned off, the distribution function decays according to

$$\frac{df}{dt} = \frac{\partial f}{\partial t} \sim -\sqrt{W} f,$$

as would be the case for obstacle losses. If we assume the same loss mechanism is present during heating as immediately after, the observed distribution function implies $\frac{dW}{dt} \sim W$, which corresponds to no simple heating model.

Lichtenberg has observed a distribution $f(W) \sim e^{-W^2}$ in an electron cyclotron heated mirror at Berkeley. Such an equilibrium distribution can arise in several ways as suggested in the table. Lichtenberg's results are also supported by numerical orbit calculations.

In the absence of any loss mechanisms,

$$\frac{\partial f}{\partial W}\frac{dW}{dt} + \frac{\partial f}{\partial t} = 0.$$

We can find a solution by separation of variables:

$$f(W, t) = f_1(W)\, f_2\, (t).$$

Substitution gives

$$\frac{f_1'\,(W)}{f_1\,(W)}\frac{dW}{dt} = -\frac{f_2'\,(t)}{f_2\,(t)} = -c_0,$$

or $f \sim e^{-W^n}\ e^{-c_0 t}$,

where $\frac{dW}{dt} \sim W^{1-n}$. For stochastic heating, $\frac{dW}{dt}$ = const. and so n = 1, giving a maxwellian distribution.

When the heating is turned off, f(W, t) is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} = -\gamma f,$$

which has the solution

$$f(W, t) = f(W, 0)\ e^{-\gamma t}.$$

For $\gamma \sim \sqrt{W}$ as observed experimentally, we conclude that $f \sim e^{-\sqrt{W}}$ as $t \to \infty$.

It appears unlikely that the observed distribution can be accounted for by any reasonable combination of heating and loss mechanisms in the above manner. The difficulty probably lies in the neglect of the diffusion term in the initial equation. In the octupole experiments, the maximum heating takes place well off the separatrix, while the measurements of f(W) and γ(W) were made on the B = 0 axis. The problem becomes quite complicated at this point, even for an isotropic distribution, because of the addition of the variable, ψ(magnetic flux function). We will propose one simple calculation that might explain the observed distribution in terms of cross field diffusion of the electrons.

Assume an equilibrium distribution on the separatrix ψ_s of the form $f(W, \psi_s, t)$ that satisfies the equation,

$$\frac{df(\psi_s)}{dt} = \frac{\partial f(\psi_s)}{\partial W} \frac{dW}{dt}\bigg|_{\psi_s} + \frac{\partial f}{\partial \psi}\bigg|_{\psi_s} \frac{d\psi}{dt}\bigg|_{\psi_s}$$

Assume obstacle losses,

$$\frac{df(\psi_s)}{dt} \sim \sqrt{W}\ f\ (\psi_s),$$

no heating on the separatrix,

$$\left.\frac{dW}{dt}\right|_{\psi_s} = 0,$$

and diffusion by convection,

$$\left.\frac{d\psi}{dt}\right|_{\psi_s} = \text{const.}$$

Finally, assume

$$f(W, \psi) \sim e^{-[W/W_0(\psi)]^n},$$

i.e., the form of the distribution is the same for all ψ, but the characteristic energy W_0 is a function ψ. Then we obtain

$$\sqrt{W}\, f \sim \left.\frac{\partial f}{\partial \psi}\right|_{\psi_s} \sim W^n f,$$

or

$$f(W, \psi_s) \sim e^{-\sqrt{W}}.$$

Note that the result is independent of the nature of the heating mechanism. The result is in agreement with experiment, but the assumptions made in the derivation are somewhat arbitrary and require further experimental verification.