Limitation of Perpetual Points for Confirming Conservation in Dynamical Systems

Sajad Jafari\(^*\) and Fahimeh Nazarimehr
Biomedical Engineering Department,
Amirkabir University of Technology,
Tehran 15875-4413, Iran
\(^*\)sajadjafari@aut.ac.ir

J. C. Sprott
Department of Physics, University of Wisconsin,
Madison, WI 53706, USA

Seyed Mohammad Reza Hashemi Golpayegani
Biomedical Engineering Department,
Amirkabir University of Technology,
Tehran 15875-4413, Iran

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Perpetual Points (PPs) have been introduced as an interesting new topic in nonlinear dynamics, and there is a hypothesis that these points can determine whether a system is dissipative or not. This paper demonstrates that this hypothesis is not true since there are counterexamples. Furthermore, we explain that it is impossible to determine dissipation of a system based only on the structure of the system and its equations.

Keywords: Perpetual point; dissipation; conservative system; multistability; coexisting attractors.

1. Introduction

Recently, many new chaotic flows have been discovered that are not associated with a saddle point, including ones without any equilibrium points, with only stable equilibria, or with a line containing infinitely many equilibrium points [Jafari & Sprott, 2013; Jafari et al., 2013; Kingni et al., 2014; Lao et al., 2014; Molaei et al., 2013; Pham et al., 2014a; Wang & Chen, 2012; 2014; Wei, 2011; Pham et al., 2014b]. The attractors for such systems have been called hidden attractors [Leonov & Kuznetsov, 2010, 2013a, 2013b, 2013c, 2014; Leonov et al., 2011a; Leonov et al., 2011b, 2012; Leonov et al., 2014; Leonov et al., 2015; Bragin et al., 2011; Kuznetsov et al., 2010; Kuznetsov et al., 2011]; and that accounts for the difficulty of discovering them since there is no systematic way to choose initial conditions except by extensive numerical search. Hidden attractors are important in engineering applications because they allow unexpected and potentially disastrous responses to perturbations in a structure like a bridge or aircraft wing. Another topic that has attracted increasing attention is multistability and coexisting attractors [Angeli et al., 2004; Pisarchik & Feudel, 2014; Blazejczyk-Okolewska & Kapitaniak, 1996, 1998; Kapitaniak, 1985; Maistrenko et al., 1997; Silchenko et al., 1999].

On the other hand, Perpetual Points (PPs) have been introduced as an interesting new topic...
2. Perpetual Points

Consider a general dynamical system
\begin{align}
  v_1 &= \dot{x}_1 = f_1(x_1, x_2, \ldots, x_n) \\
  v_2 &= \dot{x}_2 = f_2(x_1, x_2, \ldots, x_n) \\
  &\vdots \\
  v_n &= \dot{x}_n = f_n(x_1, x_2, \ldots, x_n)
\end{align}

(1)

where \( x_1, x_2, \ldots, x_n \) are dynamical variables (states), \( v_1, v_2, \ldots, v_n \) are the time derivatives of the states (velocities) and \( f_1(X), f_2(X), \ldots, f_n(X) \) are the evolution equations (velocity vectors). It is well known that the fixed points (FPs) of the above system are points \((x_1^*, x_2^*, \ldots, x_n^*)\) at which the velocities of all states are zero. We know that for all the FPs, all the accelerations are zero, since \( v_1, v_2, \ldots, v_n \) are zero. By definition [Prasad, 2015], Perpetual Points (PPs) are points like \((x_1^*, x_2^*, \ldots, x_n^*)\) at which all the accelerations are zero but the velocities are not. For more details, readers can see the original paper [Prasad, 2015].

3. Perpetual Points Cannot Determine Whether a System is Dissipative or Not

There is a hypothesis (although without mathematical proof) in [Prasad, 2015], which suggests that if a system has a PAP, then it is dissipative, and if not, it is conservative. We investigate this hypothesis using two examples:

3.1. Example one

Consider the Sprott Case A system [Silchenko et al., 1999], which is one of the oldest examples of a chaotic flow with no equilibria:

\begin{align}
  \dot{x} &= y \\
  \dot{y} &= -x + yz \\
  \dot{z} &= 1 - y^2.
\end{align}

(3)

Lyapunov exponents of this system are \((0, 0.139, 0, -0.0139)\) and the Kaplan–Yorke dimension is 3.0. This system is a special case of the Nosé–Hoover oscillator [Hoover, 1995] and describes many natural phenomena [Posch et al., 1986]. This is a conservative system, and thus it does not have attractors, but there is a chaotic sea coexisting with a set of nested tori as shown in Fig. 1.

This system has no FPs. The PPs for this system can be obtained by solving the following equations:

\begin{align}
  \dot{x} &= \ddot{y} = 0 \rightarrow -x + yz = 0 \\
  \dot{y} &= -\dot{x} + yz + y\dot{z} \\
  &= 0 \rightarrow -y^3 - xz + y^2 \\
  \dot{z} &= -y^3 + z(-x + yz) = 0 \\
  \ddot{z} &= -2y\dot{y} = 0 \rightarrow -2y(-x + yz) = 0.
\end{align}

(4)

The solution is \((0, 0, z)\), which means there is an infinite line of PPs along the \(z\)-axis. Thus the hypothesis would predict that it is dissipative, but in fact it is conservative as can be shown by numerically averaging the trace of the Jacobian matrix...
along the orbit $\langle \text{TR}(J) = (z) = 0 \rangle$ to verify that it is accurately zero.

3.2. Example two

Consider the following 2-D system:

$$\dot{x} = f(y), \quad \dot{y} = g(x). \tag{5}$$

This system is obviously conservative since the trace of the Jacobian matrix is zero. This system has a Hamiltonian of the following form:

$$H = F(y) - G(x) = \text{const} \tag{6}$$

where $F(y) = \int f(y)dy$ and $G(x) = \int g(x)dy$.

Proof. Since $\frac{\partial H}{\partial t} = \frac{\partial}{\partial y} f(y)g(x) - \frac{\partial}{\partial x} (f(y)g(x)) = 0$. \hfill \blacksquare

It is easy to make system (5) have PPs:

$$\dot{x} = \frac{\partial f}{\partial y} = g(x) \frac{\partial f}{\partial y} \quad \dot{y} = \frac{\partial g}{\partial x} = f(y) \frac{\partial g}{\partial x} \tag{7}$$

For system (5), FPs are the points for which $f(y) = g(x) = 0$. Equations (7) will be zero for these points. However, the equations are also satisfied and thus have PPs provided

$$\frac{\partial f}{\partial y} \quad \frac{\partial y}{\partial x} = 0 \quad \text{and} \quad g(x) \neq 0 \quad \text{and} \quad f(y) \neq 0. \tag{8}$$

For example, consider the system

$$\dot{x} = y^2 - 1$$
$$\dot{y} = x^2 - 1 \tag{9}$$

FPs = $(\pm 1, \pm 1)$

This means that a conservative system has PPs, which contradicts the hypothesis in [Prasad, 2015].

4. Can We Determine Whether a System is Dissipative or Not?

There are some systems of ODEs that have conservative solutions for some initial conditions and dissipative solutions for others even for all the parameters fixed. Thus, in general, it is impossible to determine whether a system is dissipative or not based only on the structure of the system and its equations, since there are cases [Sprott & Li, 2014] in which the initial conditions play an important role in the dynamics. To clarify the issue, we show some examples of systems with coexisting behaviors. Recently, Sprott, in a numerical search for chaotic systems that have no equilibrium points, discovered a simple three-dimensional time-reversible system of ODEs with quadratic nonlinearities and the property that it exhibits conservative behavior for some initial conditions and dissipative behavior for others [Sprott, 2014; Sprott et al., 2014] as given by

$$\dot{x} = y + 2xy + xz$$
$$\dot{y} = 1 - 2x^2 + yz \tag{11}$$
$$\dot{z} = x - x^2 - y^2.$$

The dissipation is given by $\text{Tr}(J) = 2(y + z)$ and the time average of $(y + z)$ is negative for some initial conditions such as $(x_0, y_0, z_0) = (2, 0, 0)$.
and zero for others such as \((x_0, y_0, z_0) = (1, 0, 0)\). The first initial condition gives a strange attractor with Lyapunov exponents \((0.0540, -0.1575)\) and a Kaplan-Yorke dimension of 2.3429, and the second initial condition gives a torus with Lyapunov exponents \((0, 0, 0)\) and a dimension of 2.0. Thus the conservative regime has quasiperiodic orbits, while the dissipative regime is chaotic. These two behaviors are shown in Fig. 2.

Another example of a system with coexisting behaviors is the thermostated harmonic oscillator where the imposed temperature field is a function of the oscillator coordinate \(x\) \cite{Sprott, Sprott et al., 2014} given by
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - yz \\
\dot{z} &= y^2 - 1 - 0.42 \tanh(x).
\end{align*}
\] (12)

This system shows interlocked phase-space behaviors, two conservative invariant tori and a dissipative limit cycle as shown in Fig. 3. The tori are produced using the initial conditions \((x_0, y_0, z_0) = (-2.3, 0, 0)\) and \((3.5, 0, 0)\) with Lyapunov exponents \((0, 0, 0)\). The limit cycle is produced using the initial conditions \((-2.7, 0, 0)\) with Lyapunov exponents \((0, -0.0256, -0.0788)\).

As these examples show, one system can have conservative and dissipative behavior for the same parameters and different initial conditions. Thus features based on the structure of the system cannot indicate (in general) whether a system is dissipative or not, although sometimes it can.

5. Conclusion
The hypothesis of the relation between PPs and dissipation of systems (which suggests that if a system has a PP then it is dissipative, and otherwise it is conservative) is not true for all cases. Furthermore, we explain that in general, it is impossible to determine dissipation of systems based only on
their structure and equations, and we clarify the issue with examples.

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References


