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Fractal Basins in the Lorenz Model

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The Lorenz mapping is a discretization of a pair of differential equations. It illustrates the pertinence of computational chaos. We describe complex dynamics, bifurcations, and chaos in the map. Fractal basins are displayed by numerical simulation.

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Nonlinear maps have contributed greatly to the understanding of complex dynamics.[1−8] An attractor of a dynamical system is a subset of the state space to which orbits originating from typical initial conditions tend as time increases. It is very common for dynamical systems to have more than one attractor. Such attractors can be static, periodic, quasiperiodic, or chaotic and are contained within a basin of attraction, which is the set of initial conditions that eventually approach the attractor. Thus the qualitative behavior of the long-time dynamics of a given system can be fundamentally different, depending upon the basin in which the initial condition lies. Such basins can vary greatly in their topological structure.

Non-uniqueness poses a challenge for predicting and controlling the dynamics in various areas of engineering and environmental sciences. Sometimes the basin boundaries are fractal sets, which can make the identification of the final behavior extremely difficult. Fractal basin boundaries have been observed and described in several models.

In this Letter, we investigate the basic patterns of complex non-uniqueness in the dynamical behavior of a class of Lorenz models proposed in Ref. [1]. The study is focused on one research area: the detection of fractal basin sets and the identification and verification of some properties of fixed points in these maps.

The basins of attraction, defining the initial conditions leading to a certain attractor, may be fractal sets. The fractal structure may be revealed by fractal basin boundaries or by patterns of self-similarity. The beauty inherent in its complex nature has become a fundamental ingredient of theory and chaotic dynamics of nonlinear systems. The analysis of these structures is useful for obtaining information about the future behavior of attractors and their basins and it provides important knowledge about the relation between them.

Fractal basin boundaries make it difficult to predict the final state of a system because the initial values can be known only to within some precision. We conclude that non-unique dynamics associated with extremely complex structures of the basin boundaries can have a profound effect on our understanding of the dynamics. Along with the references cited therein, the Lorenz model is of interest and offers a richness of bifurcations and an interesting set of dynamical phenomena due to the presence of critical or nondefinition sets.[4] The model is first investigated as a two-parameter quadratic family and its domain of fractalization is explained by using a nonclassical singularity set. The critical curve separates the plane into two regions having different numbers of real inverses (here one and three).

In this study, first we describe some peculiar properties of the Lorenz map, their dependence on the parameters and stability of the fixed points. The qualitative behavior and bifurcations of this map are examined in by using a qualitative theory and standard bifurcation theory. Then, we discuss some cases where bifurcations can lead to creation of holes in the basins of attraction and can cause qualitative changes in the structure of the domain as parameters are varied. Furthermore, critical curves are considered as a way to examine bifurcation basins.

Consider a dynamical system generated by a family of two-dimensional, continuous, noninvertible maps T_b defined by

$$T_b : \begin{cases} x' = (1+ab)x - bxy, \\ y' = (1-b)y + bx^2, \end{cases} \tag{1}$$

where a and b are real parameters, the functions $f(x,y) = (1+ab)x - bxy$ and $g(x,y) = (1-b)y + bx^2$ are continuous and differentiable and T_b is of type $Z_1 < Z_3$ whose critical curve LC_0 divides the plane into *two* areas Z_1 and Z_3 with one and three antecedents, respectively.

This map was studied by Lorenz and others cited therein. We start by giving a general summary of their results.

In Ref. [1], Lorenz studied this system obtained as an approximation to an ordinary differential equation

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by using Euler's forward differencing scheme and iterating the system of difference equations with a time increment denoted here by the parameter b. He was interested in the chaotic behavior that prevails when b is excessively large, thereby illustrating the pertinence of the concept of computational chaos. He proposed the following scenario for the breakup of an invariant circle which develops regions of increasingly sharper curvature until at a critical parameter value it develops cusps and the system thereafter exhibits chaos on an invariant set with loops.

In Ref. [8], the authors revisited this problem in more detail and showed that the invariant circle is really destroyed in a global bifurcation before the appearance of the cusps. Instead, the global unstable manifolds of saddle-type periodic points are the objects that develop cusps and subsequently loops or antennae.

Marotto[6] gave an analysis of the mapping as follows:

$$T_{a,b}:\begin{cases}x' = (ax+by)(1-ax-by),\\ y' = x.\end{cases} \tag{2}$$

He showed the existence of two distinct attractors, one resembling the quasi-linear Hénon attractor and the other consisting of points in an area. Whitehead and Macdonald[2] considered a mapping derived from a differential equation model of turbulence, given here by

$$T_{\nu,\mu,\varepsilon}:\begin{cases}x' = (1-\varepsilon\nu)x-\varepsilon xy,\\ y' = (1+\varepsilon\mu+\varepsilon x)y.\end{cases} \tag{3}$$

This mapping, as with the other models, displays chaos. For $\varepsilon = 1$, ELabbasy *et al.*[5] gave the theoretical analysis, while Tsybullin and Yudovich [3] determined interesting invariant manifolds and sets. All these maps have identical behavior and display homoclinic structure associated with the basin bifurcation between bounded and unbounded states.

The fixed points of T in Eq. (1) are solutions obtained by a trivial manipulation of Eq. (1) with $x_0 = x$ and $y_0 = y$. Besides the solution $(0;0)$, we observe that additional fixed points exist if $a > 0$. We focus attention on bifurcations playing an important role in the dynamics, those happening for $a > 0$ and $b > 0$. We can state the following proposition:

Proposition 1: If $a < 0$, then $O(0;0)$ is the unique fixed point of the map T_b defined by Eq. (1). If $a > 0$, then two further fixed points, P and P_0, exist, symmetric with respect to the y-axis, with $x = \pm\sqrt{a}$; $y = a$. Let us investigate the qualitative behaviors of the map (1). As usual, the local dynamics of map (1) in the neighborhood of a fixed point is dependent on the Jacobian matrix. The Jacobian is evaluated at the fixed point, which we denote by $J = \det DT(x,y)$. Let

$$J = \begin{vmatrix} 1+ab-by & -bx\\ 2bx & 1-b\end{vmatrix}$$

be the Jacobian matrix of T at the state variable $(x;y)$. We consider now the conditions for local stability of the fixed point $O(0;0)$ in terms of the parameters in Eq. (1). The Jacobian matrix $J_{(0,0)}$ in $O(0;0)$ has two eigenvalues $1+ab$ and $1-b$. For $b > 0$ we consider the cases with $a > 0$ and $a = 0$ and by a simple computation, it is straightforward to obtain the following result.

The fixed point $O(0;0)$ is a saddle if $b \in [0;2]$. When $a > 0$, $O(0;0)$ is a source if $b \in [2;\infty]$ and when $a > 0$, $O(0;0)$ is non hyperbolic if $a = 0$. We can see that when $b = 2$, for $a > 0$, one of the eigenvalues of $O(0;0)$ is -1 and the other is not one with module. Thus the flip bifurcation occurs with a birth of a pair of saddle-cycles of order 2.

Proposition 2: If $a = 0$, the map (1) undergoes a pitchfork bifurcation at $O(0;0)$.

Proof: By simple computation, we can prove this proposition.

Proposition 3: For $a > 0$ and $b = 2$, the map (1) undergoes a basin bifurcation between bounded and unbounded.

From the Jacobian matrix, we can see that $\det DT(x,y) = (1-b)(1+ab-by)+2b^2x^2$ vanishes on two lines given by $y = \frac{1}{b}(1+ab)+\frac{2bx^2}{(1-b)}$ for $b \neq 0$ and $b \neq 1$ or $x = 0$ for $b = 1$. The second curve given by LC_{-1}: $y = \frac{1}{b}(1+ab)+\frac{2bx^2}{(1-b)}$ (obtained by setting $J = 0$) is a curve of merging preimages and we can state as follows:

Proposition 4: For $a > 0$, $b > 0$ and $b \neq 1$, the phase plane includes a region of noninvertibility of the map (1). The noninvertibility region is an unbounded set defined by $27(1-b)^2x^2-4b(y-(1+ab)(1-b)/b)^3 < 0$. This curve possesses a pointing cusp on the y-axis at $x = 0$. The antecedents have coordinates (x,y) such that from map (1), x satisfies $b^2x^3-bx(y'-(1+ab)(1-b))/b)-(1-b)x' = 0$ and $y = ((1+ab)x-x')/bx$. Moreover, the map T has $Z1 < Z3$ whose critical curve is LC: $27((1-b)^2x^2-4b(y-(1+ab)(1-b)/b)^3 = 0$ with an image LC_{-1} that separates the plane into two areas $Z1$ and $Z3$ where there exist one and three antecedents, respectively. We now consider the conditions for asymptotic stability of the fixed point P(resp. P_0) in terms of the parameters of Eq. (1). P is a nontrivial fixed point. Under certain conditions, the map (1) also undergoes a flip bifurcation at P as shown with $J_{(\pm\sqrt{a},a)} = \lambda^2+\lambda(b-2)+2b^2a-b+1$. Let λ_1 and λ_2 be the two real roots, we have: (i) $P(\sqrt{a},a)$ is a sink if one of the following conditions holds: $0 < b < \frac{4}{1+\sqrt{1-8a}}$, $0 < a < \frac{1}{8}$, or $b < 4$, $a = \frac{1}{8}$. (ii) $P(\sqrt{a},a)$ is a saddle if $a \in [0;\frac{1}{8}]$ and b covering $[\frac{4}{1+\sqrt{1-8a}};\frac{4}{1-\sqrt{1-8a}}]$. (iii) $P(\sqrt{a},a)$ is a source if one of the following conditions holds: $\frac{4}{1-\sqrt{1-8a}} < b$, $0 < a < \frac{1}{8}$, or $b > 4, a = \frac{1}{8}$. For $a > \frac{1}{8}$, and $b = \frac{1}{2a}$ we can obtain that the eigenvalues of P(resp. P_0) are a pair of conjugate complex numbers with module one. From the above, one can draw the following conclusion.

Proposition 5: If $b = \frac{1}{2a}$, the map (1) undergoes a Neimark–Sacker bifurcation at $P(\sqrt{a}, a)$. Consider the case $b > 1$ as shown in Fig. 1. For $b = 1.5$, we have two chaotic attractors A_1 and A_2 whose immediate basins are delimited by the stable manifold of the saddle fixed point $O(0,0)$. The right and left branches of this unstable manifold converge toward the two chaotic attractors and are dense inside them. We can see in Fig. 1 that the immediate basin of A_1 (resp. A_2) has a hole D_2 (resp. D_1) which is a connected component of the basin of A_2. The islands D_i, $i = 1, 2$ belong to the area Z_1.

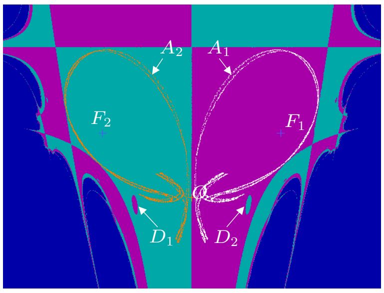

Fig. 1. Attractors A_1 and A_2 for $a = 0.5$, $b = 1.5$.

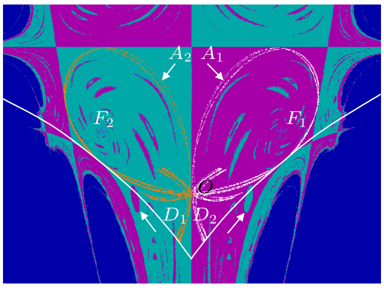

Fig. 2. Islands converge to F_i and D_i cross LC, $a = 0.5$, $b = 1.513$.

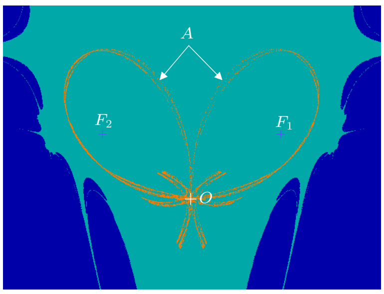

Fig. 3. Homoclinic bifurcation ($1.516 < b = b^* < 1.517$).

For $b = 1.513$, Fig. 2 shows that the island D_1 (resp. D_2) has entered inside Z_3 and then we obtain a sequence of preimages of D_1 (resp. D_2) converging toward the unstable focus F_2 (resp. F_1). Since the boundaries of D_i and their preimages are the connected components of the stable manifold of the saddle $O(0,0)$, an infinity of heteroclinic orbits appears to connect $O(0,0)$ and F_1 (resp. F_2). Each orbit contains a point belonging to the boundary of D_1 (resp. D_2).

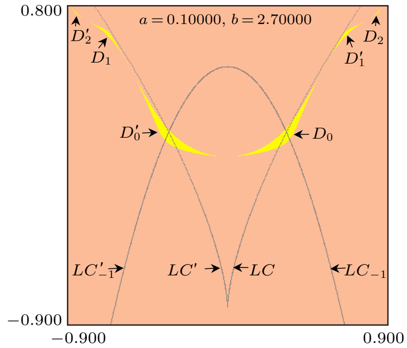

Fig. 4. First tangential contact between LC and the island D_1'.

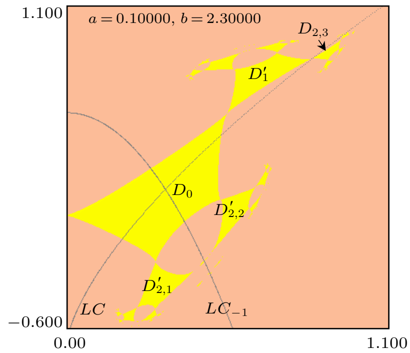

Fig. 5. Other bifurcations which give an arborescent sequence of islands.

For $b = 1.517$, Fig. 3 shows that the two attractors have merged into a unique chaotic attractor A. This merger coincides with a homoclinic bifurcation of the saddle point $O(0,0)$ that occurs for $1.516 < b = b^* < 1.517$. A plot of the fractal basins associated with a dynamical system provides a qualitative indication of the difficulty in predicting its future evolution. Since the relation between fractacality and nonlinear dynamics has been established, we use a numerical technique to characterize the fractal nature of the basins. For the case $a = 0.1$ and varying the parameter b, there are two stable fixed points of the node type, each with an unconnected basin. We observe that the island number increases when b decreases. For $b = 2.7$, there is a first tangential contact between LC and D_1' (the yellow island in Fig. 4). For $b = 2.35$, a new tangential contact between LC and D_2 occurs as shown in Fig. 5, after which each island is divided in two islands. Therefore, there are many bifurcations that change the number of islands, beginning at $b = 2.7$. This kind of bifurcation repeats several times and at the end, the basin is strongly fractal.

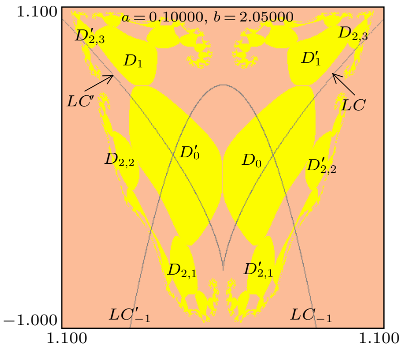

Fig. 6. Fractal basin.

Basins constitute an interesting object of study by themselves. The strong dependence on parameters generates a rich variety of complex patterns on the plane and gives rise to different types of basin fractalization as a consequence, such as contact bifurcations between a critical curve segment and the basin boundary. Taking into account the complexity of the matter and its nature, the study of these phenomena can be carried out only by numerical investigation guided by fundamental considerations as found in Ref. [4].

For $b = 2.05$, Fig. 6 shows that we are close to the bifurcation value at $b = 2$ that leads to basin transformation from a bounded basin to an unbounded basin for $b < 2$. The saddle fixed point at $O(0, 0)$ has become an unstable node, with the birth of a pair of 2-cycles of the saddle type, symmetric with respect to the y-axis. Lorenz predicted chaotic behavior in his map model. His paper is a milestone in the study of deterministic nonlinear dynamical systems. This fact has fundamental and known consequences. We clarify the concept of fractacality by evaluating and plotting basins computationally.

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