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Classification of three-dimensional quadratic diffeomorphisms with constant Jacobian

Abstract The 3-D quadratic diffeomorphism is defined as a map with a constant Jacobian. A few such examples are well known. In this paper, all possible forms of the 3-D quadratic diffeomorphisms are determined. Some numerical results are also given and discussed.

Keywords 3-D quadratic diffeomorphism, classification, chaos

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1 Introduction

The most general 3-D quadratic map is given by

\[
f(x, y, z) = \begin{pmatrix} a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 xz + a_9 yz \\ b_0 + b_1 x + b_2 y + b_3 z + b_4 x^2 + b_5 y^2 + b_6 z^2 + b_7 xy + b_8 xz + b_9 yz \\ c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 y^2 + c_6 z^2 + c_7 xy + c_8 xz + c_9 yz \end{pmatrix}
\]

(1)

where \((a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}\) are the bifurcation parameters. Polynomial maps of the form (1) are part of the models of storage ring elements in the “thin lens” approximation [21]. In this case the one-turn map is difficult to evaluate with both speed and accuracy, because a modern storage ring consists of \(10^3\) to \(10^4\) elements.

The occurrence of hyperchaotic and wild-hyperbolic attractors in some 3-D quadratic maps of the form (1) [3, 5, 7–11] makes them useful in potential applications [4, 6]. If the map (1) has constant Jacobian, then it is a 3-D quadratic diffeomorphism. Some possible generalizations of the 2-D Hénon map were introduced in [7–11]. Note that the attractors obtained from these selected generalizations are very similar to the Lorenz and Shimizu-Morioka attractors [1, 2]. Some forms of 3-D quadratic diffeomorphisms are important because they have some relation to Lorenz attractors [1–3, 7], to the study of unfolding 2-D maps to maps of higher dimension, and to the study of homoclinic phenomena in dynamical systems [8, 9], among others.

However, some forms of the 3-D quadratic diffeomorphisms are well known [3–5, 7–11], especially those with a quadratic inverse, where it was shown in Ref. [8] that any 3-D quadratic diffeomorphism with a quadratic inverse and constant Jacobian can be written in a normal form. However, this result does not give any information about the shapes of this type of map. Therefore, in this paper we investigate all the possible forms of the 3-D quadratic diffeomorphisms, especially, those without quadratic inverse, i.e., with an inverse of higher degree.
contains at least one quadratic nonlinearity. Some of the simplest possible cases are presented here and discussed:

\[
J(x, y) = \begin{pmatrix}
   a_1 + 2a_4x + a_7y + a_8z & a_2 + 2a_5y + a_7x + a_9z & a_3 + a_8x + 2a_9y + a_{10}
   
   b_1 + 2b_4x + b_7y + b_8z & b_2 + 2b_5y + b_7x + b_9z & b_3 + b_8x + 2b_9y + b_{10}
   
   c_1 + 2c_4x + c_7y + c_8z & c_2 + 2c_5y + c_7x + c_9z & c_3 + c_8x + 2c_9y + c_{10}
\end{pmatrix}
\]

(2)

The determinant of the Jacobian matrix (2) of the map (1) is given by

\[
\det J(x, y, z) = \psi_1 + \Phi_1(x, y, z) + \Phi_2(x, y, z)
   + \Phi_3(x, y, z)
\]

(3)

where

\[
\Phi_1(x, y, z) = \xi_1x + \xi_2y + \xi_3z + \xi_4xy + \xi_5xz
   + \xi_6yz + \xi_7xyz + \xi_8x^2
\]

\[
\Phi_2(x, y, z) = \xi_9y^2 + \xi_{10}z^2 + \xi_{11}x^2 + \xi_{12}yz^2
   + \xi_{13}z^2 + \xi_{14}x + \xi_{15}y + \xi_{16}z + \xi_{17}x + \xi_{18}y + \xi_{19}z
\]

(4)

\[
\Phi_3(x, y, z) = \xi_{20}y^3 + \xi_{21}z^3 + \xi_{22}x^2 + \xi_{23}y^2 + \xi_{24}z^2 + \xi_{25}x + \xi_{26}y + \xi_{27}z
\]

(5)

The method of analysis is the rigorous computation of the Jacobian of the map (1), and hence we determine sufficient conditions for the 3-D quadratic diffeomorphisms in the sense that their Jacobian is a constant and contains at least one quadratic nonlinearity. Some of the simplest possible cases are presented here and discussed.

2 Sufficient conditions for the 3-D quadratic diffeomorphisms

In this section, we will find sufficient conditions for the 3-D quadratic diffeomorphisms. Indeed, the Jacobian matrix of the map (1) is given by

\[
\psi_7 = -a_3b_2c_7 + a_3c_2b_7 - b_2c_1a_9 + b_2c_3a_7 - b_3c_2a_7
\]

\[
\psi_8 = 2a_1c_2b_6 - 2a_1b_2c_6 + 2a_2b_1c_6 - 2a_2c_1b_6
   - 2b_1c_2a_6 + 2b_2c_1a_6
\]

\[
\psi_9 = a_1b_2c_9 - a_1c_2b_9 - a_2b_3c_8 + a_2c_3b_8 - b_1a_3c_9
   + b_1c_3a_9
\]

\[
\psi_{10} = a_3c_1b_9 - a_3b_2c_9 - b_2c_1a_9 - c_1b_3a_9 + b_3c_2a_9 + a_3b_2c_9
\]

\[
\psi_{11} = 4a_3b_4c_5 - 4a_3b_5c_4 - 4a_4b_3c_5 + 4a_4c_3b_5
   + 4b_3a_5c_4 - 4a_5b_3c_4
\]

\[
\psi_{12} = 2a_1b_5c_8 - 2a_1c_5b_8 - 2b_1a_5c_8 + 2b_1c_5a_8
   + 2c_1a_5b_8 - 2c_1b_5a_8
\]

\[
\psi_{13} = -2a_2b_4c_9 + 2a_2c_4b_9 - 2b_2a_4c_9 - 2b_2c_4a_9
   - 2a_4c_2b_9 + 2b_4a_2c_9
\]

\[
\psi_{14} = a_1b_7c_9 - a_1c_7b_9 - a_2b_7c_8 + a_2b_5c_7 + a_1b_6c_9
   + b_1a_9c_7 + b_2a_7c_9
\]

\[
\psi_{15} = -b_2a_8c_7 + c_1a_7b_9 - c_1b_7a_9 - c_2a_7b_8 + c_2a_8b_7
\]

\[
\psi_{16} = 4a_2b_6c_7 - 4a_2c_6b_7 - 4b_2a_6c_7 + 4b_2c_6a_7
   + 4a_4c_2b_9 - 4c_4a_2b_9
\]

(7)

\[
\psi_{17} = 2a_1b_6c_7 - 2a_1b_6c_7 - 2b_1a_6c_7 + 2b_1a_7c_6
   + 2c_1a_6b_9 - 2c_1a_9b_6
\]

\[
\psi_{18} = -2a_3b_6c_9 + 2a_3c_6b_9 - 2a_4b_6c_9 - 2a_4c_6b_9
   + 2b_4c_6a_9 - 2b_4a_6c_9
\]

\[
\psi_{19} = a_1b_3c_9 - a_1b_3c_9 - b_1a_3c_9 + b_1a_9c_3 + a_3b_7c_8
   - a_3b_8c_7
\]

\[
\psi_{20} = c_1a_9b_9 - c_1a_9b_9 - b_3a_7c_8 + b_3a_8c_7 + c_3a_7b_8
   - c_3a_8b_7
\]

\[
\psi_{21} = 4a_1b_5c_6 - 4a_1b_5c_6 + 4b_1a_5c_6 - 4b_1a_5c_6
   - 4c_1a_5b_6
\]

\[
\psi_{22} = 4c_1a_5b_6 - 2a_2b_6c_7 + 2a_2b_7c_6 + 2b_2a_6c_7
   - 2b_2a_7c_6
\]

\[
\psi_{23} = -2c_2a_9b_7 + 2a_5c_3b_8 - b_2a_9c_9 + c_3b_7a_9
\]

\[
\psi_{24} = 2c_2a_9b_6 - 2a_3b_5c_8 - 2a_3c_5b_8 - 2b_3a_5c_8
   + 2b_3c_5a_8
\]
\[
\psi_{25} = -2a_3b_5a_8 + a_2b_8c_9 - a_2b_9c_8 - a_3b_7c_9 \\
+ a_3c_7b_9 \\
\psi_{26} = b_2a_9c_8 + b_3a_7c_9 - b_3a_9c_7 + c_2a_8b_9 - c_2a_9b_8 \\
- c_3a_7b_9 \\
\psi_{27} = 4a_1b_6c_5 - 4a_1b_5c_6 + 4a_5b_6c_4 - 4a_5c_4b_6 \\
- 4b_3a_4c_5 + 4a_5b_4c_4 \\
\psi_{28} = 27b_8c_9 - a_2b_9c_8 - a_2b_7c_9 + a_8c_7b_9 + b_7a_9c_8 \\
- a_9b_8c_7 \\
\psi_{29} = 2a_2c_2b_8 - 2a_2b_4c_8 + 2a_3b_4c_7 - 2a_3c_4b_7 \\
+ 2b_2a_4b_8 - b_2c_4a_8 \\
\psi_{30} = -2a_2b_8c_7 + 2a_3c_2b_7 + 2a_4c_1a_7 \\
+ 2c_5b_4a_8 - 2b_4c_3a_7 \\
 Buddha}
(8)

\[
\psi_{31} = a_1b_7c_8 - a_1b_8c_7 - b_1a_7c_8 + b_1a_8c_7 + c_1a_7b_8 \\
- c_1a_8b_7 \\
\psi_{32} = 2a_1c_6b_9 - 2a_1b_9c_9 + 2b_1a_5c_9 - 2b_1c_5a_9 + 2a_3b_5c_7 \\
\psi_{33} = -2a_3c_5b_7 - 2a_1b_5a_8 + 2c_1b_5a_9 - 2b_3a_5c_7 \\
+ 2a_3b_6c_5 + 2a_5c_5b_7 \\
\psi_{34} = -2b_3b_8a_7 + a_2b_7c_9 - a_2c_7b_9 - b_2a_9c_9 \\
+ b_2a_9c_7 + c_2a_2b_9 - c_2b_7a_9 \\
\psi_{35} = 2a_1c_5b_9 - 2a_1b_9c_9 + 2b_1a_6b_9 - 2a_2b_6c_8 + b_2a_6c_9 - 2b_4c_3a_9 \\
\psi_{36} = 2b_3b_8a_7 + 2a_3b_8c_6 + 2a_1b_9b_9 + 2b_1b_9c_9 + 2c_1b_9a_9 \\
+ 2c_2b_9b_8 - 2c_2b_9a_8 \\
\psi_{37} = a_3b_7c_9 - a_3b_9c_8 + b_3a_8c_9 + b_3a_9c_8 + c_3a_9b_8 \\
- c_3a_9b_8 \\
\psi_{38} = a_3b_6c_7 - a_3b_9c_8 + b_4a_7c_9 - b_4a_8c_9 + c_4a_7b_8 \\
+ c_4a_8b_7 \\
\psi_{39} = a_5c_7b_9 - a_5b_9c_9 + b_5a_7c_9 - b_5a_9c_7 + c_5b_9a_9 \\
+ c_5b_9b_9 \\
\psi_{40} = a_5b_7c_9 - a_5b_9c_8 + b_5a_8c_9 + a_5c_8b_9 \\
+ a_5c_8b_9 \\
\psi_{41} = a_4c_5b_8 - a_4b_5c_9 + a_5b_4c_8 - a_5c_4b_9 - b_4c_5a_8 \\
+ b_5c_4a_8 \\
\psi_{42} = 2a_4b_5b_9 - 2a_4c_5b_9 - 2a_5b_4c_9 + 2a_5c_4b_9 \\
+ 2b_4c_5a_9 - 2b_5c_4a_9 \\
\psi_{43} = -2a_5b_7c_8 + a_5b_9c_7 + b_5a_7c_9 - b_5a_9c_7 \\
- a_5b_7c_9 + c_5a_8b_7 \\
\psi_{44} = a_4b_7c_9 - a_4b_8c_7 + b_4a_6c_7 - b_4a_7c_6 - a_4c_1b_7 \\
+ c_4a_7b_6 \\
\psi_{45} = 2a_4b_9c_8 - 2a_4c_6b_9 - 2b_4a_6c_9 + 2b_4c_6a_9 \\
+ 2a_4b_5c_9 - 2b_4c_6a_9 \\
\psi_{46} = a_6b_7c_9 - a_6b_8c_7 - a_7b_9c_8 + a_7c_6b_9 + b_6a_9c_7 \\
- a_8b_7c_9 \\
\psi_{47} = a_4c_7b_9 - a_4b_7c_9 + b_4a_7c_9 - b_4a_9c_7 - c_4a_7b_9 \\
+ c_4a_7b_9 \\
\psi_{48} = a_4b_8c_9 - b_4a_8c_9 - b_4a_9c_9 - c_4a_8b_9 \\
+ c_4a_8b_9 \\
\psi_{49} = 2a_5b_7c_9 - 2a_5b_9c_7 - 2a_6b_5c_7 - 2a_6c_5b_7 \\
- 2b_5a_7c_9 + 2a_7b_5c_7 \\
\psi_{50} = a_5b_7c_8 + a_5b_9c_9 - b_5a_8c_9 + b_5a_9c_8 + c_5a_8b_9 \\
- c_5a_8b_9 \\
\psi_{51} = 2a_5b_8c_9 - 2a_5b_9c_8 + 2a_6b_5c_8 + 2a_6c_5b_8 \\
+ 2b_5a_8c_9 - 2b_6c_5a_8 \\
\psi_{52} = a_8b_7c_9 - a_6c_7b_9 - a_7b_9c_8 + a_7c_6b_9 + b_6a_9c_7 - b_7c_6a_9 \\
(9)

Hence the map (1) is a 3-D quadratic diffeomorphism if and only if

\[
\begin{cases}
\psi_1 \neq 0 \\
\psi_1(x, y, z) = 0 \\
\psi_1(x, y, z) = 0 \\
\end{cases}
\]\(\text{for all } (x, y, z) \in \mathbb{R}^3\).

This is possible if \(\xi_i = 0, i = 1, \ldots, 21\), i.e.,

\[
\psi_j(a_i, b_i, c_i)_{1 \leq i \leq 9} = 0, \quad j = 2, \ldots, 52
\]\(\text{(12)}\)

Because \(\det J(x, y, z)\) is a polynomial function, the only possible case for a non-vanishing constant determinant is when (11) and (12) hold. We define the following subsets of \(\mathbb{R}^{27}\):

\[
\Omega_1 = \left\{(a_i, b_i, c_i)_{1 \leq i \leq 9} \in \mathbb{R}^{27} : \psi_1(a_i, b_i, c_i)_{1 \leq i \leq 9} \neq 0 \right\}
\]

\[
\Omega_2 = \left\{(a_i, b_i, c_i)_{1 \leq i \leq 9} \in \mathbb{R}^{27} : \psi_1(a_i, b_i, c_i)_{1 \leq i \leq 9} = 0 \right\}
\]

\(j = 2, \ldots, 52\)
(13)

Thus if there are vectors \((a_i, b_i, c_i)_{1 \leq i \leq 9} \in \mathbb{R}^{27}\) such that \((a_i, b_i, c_i)_{1 \leq i \leq 9} \neq (0, 0, \ldots, 0)\), i.e., \(j=1 \Omega_j \neq \emptyset\), then \(\det J(x, y, z)\) is a non-zero constant for all \((x, y, z) \in \mathbb{R}^3\). However, the system of equations (13) can be rewritten as

\[
\psi_1 \left((a_i, b_i, c_i)_{1 \leq i \leq 9}\right) \neq 0
\]

\[
AC = O
\]\(\text{(14)}\)

where \(A = A \left((a_i, b_i, c_i)_{1 \leq i \leq 9}\right)\) is a 51 \times 9 matrix, \(C = (c_i)_{1 \leq i \leq 9}\) is a 9 \times 1 vector of unknowns, and \(O\) is the null vector of \(\mathbb{R}^{51}\). The classical method of Fontené-Rouché can be used to solve this system of equations by
introducing the so-called principal determinant. However, this is very hard to do theoretically since it is difficult to determine the set of principal unknowns.

Thus, we have proved the following theorems:

**Theorem 1** The map (1) is a 3-D quadratic diffeomorphism if and only if \( \Omega = \bigcap_{j=1}^{52} \Omega_j \neq \emptyset \).

**Theorem 2** The set \( \Omega = \bigcap_{j=1}^{52} \Omega_j \) contains at least one 3-D quadratic diffeomorphism.

**Proof** The well known generalized Hénon map given in Ref. [5] satisfies all the above conditions.

Note that the existence of an inverse is guaranteed by the so called real Jacobian conjecture introduced by O.T. Keller in 1939 [13, 14], and also the upper bound for the degree of the inverse of a quadratic map on \( \mathbb{R}^n \) is known to be \( 2^{n-1} \) [14], in our case the upper bound is 4. On the other hand, it was shown in Ref. [8] that any 3-D quadratic diffeomorphism with a quadratic inverse and constant Jacobian can be written in the following form:

\[
g(x, y, z) = \begin{pmatrix} y \\ z \\ d_0 + d_1 x + d_2 y + d_3 z + d_4 y^2 + d_5 z^2 + d_6 yz \end{pmatrix}
\]

where \( d_i \) are the bifurcation parameters.

Therefore, we classify all the 3-D quadratic diffeomorphisms into two classes: those with a quadratic inverse, and those with no quadratic inverse, where we determine exactly all the possible forms of these two families. Indeed, as a test of the previous analysis, and for the sake of simplicity and without loss of generality, we can assume that

\[
a_0 \neq 0, a_1 = a_3 = 0 \\
b_0 = b_1 = b_2 = 0 \\
c_0 = c_2 = c_3 = 0 \\
c_1 \neq 0, a_2 \neq 0, b_3 \neq 0
\]

Then one has the following conditions:

\[
a_5 = 0, \quad a_7 = 0, \quad a_9 = 0, \quad b_0 = 0, \quad b_8 = 0 \\
b_8 = 0, \quad c_4 = c_7 = -\frac{c_1 b_0}{b_3}, \quad c_8 = 0 \\
a_1 b_9 = 0, \quad a_4 c_5 = 0, \quad a_4 c_9 = 0, \quad a_4 b_7 c_6 = 0 \\
a_8 b_5 = 0, \quad a_8 c_5 = 0, \quad a_8 b_7 = 0, \quad a_8 c_9 = 0, \quad a_8 b_9 = 0 \\
a_6 b_7 = 0, \quad a_6 b_9 = 0, \quad a_6 b_4 c_9 = 0 \\
c_6 b_5 = 0, \quad c_6 b_7 = 0 \\
c_1 a_6 b_5 + a_2 b_4 c_6 = 0, \quad b_4 c_9 = 0 \\
a_4 b_5 c_6 + b_4 a_6 c_5 = 0
\]

Then one has the following conditions:

\[
c_1 b_9^2 + b_3 b_7 c_9 = 0
\]

Therefore, one of the possible forms of the 3-D quadratic diffeomorphisms is given by

\[
f(x, y, z) = \begin{pmatrix} a_0 + a_2 y + a_4 x^2 + a_6 z^2 + a_8 xz \\ b_1 z + b_4 x^2 + b_5 y^2 + b_7 xy + b_9 yz \\ c_1 x + c_3 y^2 - \frac{c_1 b_9}{b_3} xy + c_9 yz \end{pmatrix}
\]

with the conditions (16) and (17). More analysis on the conditions (17) show the existence of more than 71 cases. Some of them are listed below, especially those with one and two nonlinearities. On the other hand, we say that the maps \( f \) and \( g \) given respectively by (15) and (18) are affinely conjugate if there exists an affine transformation \( h \) such that

\[
g \circ h(x, y, z) = h \circ f(x, y, z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3
\]

As a result if the map (18) is affinely conjugate to the map (15), then the map (18) has quadratic inverse. We use this remark to characterize the 3-D quadratic diffeomorphisms without quadratic inverse. Indeed, the transformation \( h \) is defined by

\[
h(x, y) = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}
\]

with the condition of invertibility given by

\[
d = (h_{22} h_{33} - h_{23} h_{32}) h_{11} + (h_{31} h_{23} - h_{21} h_{33}) h_{12} + (h_{21} h_{32} - h_{22} h_{31}) h_{13} \neq 0
\]

Such affine transformation \( h \) exists if and only if

\[
E = \bigcap_{i=0}^{i=30} E_i \neq \emptyset
\]

where

\[
E_0 : d = -3a_2 c_1 b_3 h_{11} h_{12} h_{13} + a_2^2 b_3 h_{11}^3 + c_1 b_9^2 h_{12}^3 + a_2 c_7 h_{13}^3 \\
E_1 : s_1 = s_3 - a_0 h_{11} - a_0 c_1 h_{13} \\
E_2 : s_2 = s_3 - a_0 c_1 h_{13} \\
E_3 : a_8 h_{11} = 0 \\
E_4 : a_6 h_{11} = 0 \\
E_5 : a_6 h_{13} = 0 \\
E_6 : a_8 h_{13} = 0 \\
E_7 : h_{22} = a_2 h_{11} \\
E_8 : h_{21} = c_1 h_{13}
\]
\[ E_0 : h_{23} = b_3 h_{12} \]
\[ E_{10} : h_{32} = c_1 a_2 h_{13} \]

and

\[ E_{11} : h_{31} = c_1 b_3 h_{12} \]
\[ E_{12} : h_{33} = b_3 a_2 h_{11} \]
\[ E_{13} : b_3 h_{12} + c_1 h_{13} = 0 \]
\[ E_{14} : a_2 h_{11} + b_3 h_{12} = 0 \]
\[ E_{15} : b_3 h_{12} + c_1 h_{13} = 0 \]
\[ E_{16} : b_7 h_{12} - \frac{c_1}{c_3} b_3 h_{13} = 0 \]
\[ E_{17} : a_2 b_3 h_{11} + b_3 c_3 h_{12} = 0 \]
\[ E_{18} : a_2 b_3 h_{11} + b_3 c_3 h_{12} = 0 \]
\[ E_{19} : a_2 b_3 h_{11} - c_1 b_3 h_{12} = 0 \]
\[ E_{20} : a_2 b_3 h_{11} + c_1 a_2 h_{13} = 0 \]

and

\[ E_{21} : \zeta_1 + \zeta_2 = 0 \]
\[ E_{22} : b_3 h_{11} - b_7 h_{13} + b_3 d_3 h_{11} h_{12} + 2d_5 h_{11} h_{13} + 2c_1 b_5 d_3 h_{12} h_{13} + c_1 d_5 h_{13}^2 = 0 \]
\[ E_{23} : -c_1 d_6 h_{12} + a_2 b_3 d_3 h_{11} h_{12} + a_2 b_3 d_3 h_{11}^2 + b_5 d_3 h_{12}^2 = 0 \]
\[ E_{24} : -a_2 b_3 h_{12} - a_2 b_3 h_{13} + c_1 b_3 d_5 h_{12} h_{13} + c_1 b_3 d_5 h_{12}^2 + c_1 d_5 h_{13}^2 = 0 \]
\[ E_{25} : -b_3 c_3 h_{11} - c_1 b_3 h_{12} + a_2 c_1 d_5 h_{11} h_{13} + a_2 c_1 d_5 h_{11}^2 + a_2 c_1 d_5 h_{12}^2 = 0 \]
\[ E_{26} : \zeta_3 + \zeta_4 = 0 \]
\[ E_{27} : 2d_5 h_{12} h_{13} - a_3 h_{12} + a_2 d_3 h_{11} h_{13} + 2a_2 b_3 d_3 h_{11} h_{12} + b_5 d_3 h_{12}^2 = 0 \]
\[ E_{28} : \zeta_5 + \zeta_6 = 0 \]
\[ E_{29} : \zeta_7 + \zeta_8 = 0 \]
\[ E_{30} : \zeta_9 + \zeta_{10} = 0 \]

where

\[ \zeta_1 = -b_3 c_3 h_{11} - c_1 b_3 h_{13} + 2b_3 d_5 h_{11} h_{12} \]
\[ \zeta_2 = a_2 c_1 b_3 d_3 h_{11} h_{13} + c_1 b_3 d_3 h_{12} h_{13} + a_2 b_3 d_3 h_{12}^2 \]
\[ \zeta_3 = d_0 + (d_1 + d_2 + d_3 - 1) s_3 - a_0 c_1 b_3 h_{12} - a_0 d_1 h_{11} - c_1 a_0 (2d_5 + d_6) h_{13} s_3 \]
\[ \zeta_4 = -c_1 a_0 (d_1 + d_2) h_{13} + (d_5 + d_6 + d_9) s_3^3 \]
\[ + a_2 d_5 h_{13}^2 \]
\[ \zeta_5 = d_1 h_{11} + c_1 d_3 h_{13} - a_2 c_1 b_3 h_{11} + c_1 b_3 d_3 h_{12} + c_1 (2d_5 + d_9) h_{13} s_3 \]
\[ \zeta_6 = b_3 c_1 (2d_6 + d_9) h_{12} s_3 - a_0 c_1^2 b_3 d_3 h_{12} h_{13} - 2a_0 c_1^2 d_5 h_{13}^2 \]
\[ \zeta_7 = -a_0 a_2 c_1^2 d_5 h_{13} + (a_2 c_1 d_3 - 2a_0 a_2 c_1 d_5) h_{13} \]
\[ + s_3 (a_2 c_1 d_3 + a_2 c_1 d_5) h_{13} \]
\[ \zeta_8 = d_1 h_{12} + a_2 d_3 h_{11} - a_2 c_1 b_3 h_{12} + s_3 (2a_0 d_5 h_{11} + a_2 d_5 h_{11}) \]
\[ \zeta_9 = a_2 b_3 d_3 h_{11} + b_3 d_3 h_{12} + (d_1 - a_2 c_1 b_3) h_{13} + b_3 (2d_5 + d_9) h_{12} s_3 \]
\[ \zeta_{10} = b_3 (2d_6 + d_9) h_{11} s_3 - 2a_0 c_1 b_3 d_3 h_{12} h_{13} - a_0 a_2 c_1 b_3 d_3 h_{12} h_{13} \]

Thus the transformation \( h \) takes the form:

\[
\begin{bmatrix}
    h_{11} & h_{12} & h_{13} \\
    c_1 b_3 h_{11} & a_2 b_3 h_{11} & b_3 a_2 h_{11} \\
    c_1 b_3 h_{12} & c_1 a_2 h_{12} & b_3 a_2 h_{11}
\end{bmatrix}
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
\]

(25)

We remark that If \( (a_i, b_i, c_i)_{0 \leq i \leq 9} \in \tilde{E} = \bigcup_{i=0}^{i=20} \tilde{E}_i \), where \( \tilde{E} \) is the compliment of \( E \), then maps (15) and (18) are not affine conjugate. For example if \( a_0 \neq 0 \) or \( a_8 \neq 0 \), i.e. \( (a_i, b_i, c_i)_{0 \leq i \leq 9} \in \tilde{E}_1 \cup \tilde{E}_2 \) then one has that \( h_{11} = 0 \) and \( h_{13} = 0 \), then if \( b_4 \neq 0 \) or \( b_5 \neq 0 \) or \( b_7 \neq 0 \), or \( b_9 \neq 0 \), or \( c_5 \neq 0 \) or \( c_7 \neq 0 \), then \( h_{12} = 0 \), then the transformation \( h \) is not invertible. Thus, a characterization of 3-D quadratic diffeomorphisms without quadratic inverse is given in the following Theorem:

**Theorem 3** If \( (a_i, b_i, c_i)_{0 \leq i \leq 9} \in \tilde{E} = \bigcup_{i=0}^{i=20} \tilde{E}_i \), then the map (18) has no quadratic inverse.

On the other hand, and regarding the conditions (17), it is important to note that the 3-D quadratic diffeomorphism can be written with several nonlinearities, for example all maps with one nonlinearity are given by

\[
\begin{bmatrix}
    f(x, y, z) = \begin{pmatrix}
        a_0 + a_2 y + a_4 x^2 \\
        b_3 z \\
        c_1 x
    \end{pmatrix}
\end{bmatrix}
\]

(28)

\[
\begin{bmatrix}
    f(x, y, z) = \begin{pmatrix}
        a_0 + a_2 y + a_6 x^2 \\
        b_3 z \\
        c_1 x
    \end{pmatrix}
\end{bmatrix}
\]

(29)

\[
\begin{bmatrix}
    f(x, y, z) = \begin{pmatrix}
        a_0 + a_2 y + a_8 x z \\
        b_3 z \\
        c_1 x
    \end{pmatrix}
\end{bmatrix}
\]

(30)
with two nonlinearities are given by the other hand, all the 3-D quadratic diffeomorphisms (36) are conjugate to the dynamics of the map (15). On verse.

Theorem 4

Any 3-D quadratic diffeomorphism of the family (18) with one nonlinearity has a quadratic inverse.

Therefore, the dynamics of all the cases from (28) to (36) are conjugate to the dynamics of the map (15). On the other hand, all the 3-D quadratic diffeomorphisms with two nonlinearities are given by

\[ f(x, y, z) = \begin{pmatrix} a_0 + a_2y + a_4x^2 + a_6z^2 \\ b_3z \\ c_1x \end{pmatrix} \quad (37) \]

\[ f(x, y, z) = \begin{pmatrix} a_0 + a_2y + a_4x^2 + a_8xz \\ b_3z \\ c_1x \end{pmatrix} \quad (38) \]

\[ f(x, y, z) = \begin{pmatrix} a_0 + a_2y + a_4x^2 \\ b_3z + b_4x^2 \\ c_1x \end{pmatrix} \quad (39) \]

\[ f(x, y, z) = \begin{pmatrix} a_0 + a_2y + a_4x^2 \\ b_3z + b_5y^2 \\ c_1x \end{pmatrix} \quad (40) \]

\[ f(x, y, z) = \begin{pmatrix} a_0 + a_2y \\ b_3z + b_4x^2 + b_7xy \\ c_1x \end{pmatrix} \quad (53) \]

After some tedious calculations, we have proved the following theorem:

**Theorem 4** Any 3-D quadratic diffeomorphism of the family (18) with one nonlinearity has a quadratic inverse.

Therefore, the dynamics of all the cases from (28) to (36) are conjugate to the dynamics of the map (15). On the other hand, all the 3-D quadratic diffeomorphisms with two nonlinearities are given by
An example of a 3-D quadratic diffeomorphism with three nonlinearity is given by:
\[
f(x, y, z) = \begin{pmatrix} a_0 + a_2y + a_4x^2 + a_6xz \ 
  b_3z \ 
  c_1x \end{pmatrix}
\]
and it has a quadratic inverse and thus is conjugate to the map (15). Also, an example of those with four nonlinearity is given by:
\[
f(x, y, z) = \begin{pmatrix} a_0 + a_2y + a_6z^2 \ 
  b_3z \ 
  c_1x + c_5y^2 + c_9yz \end{pmatrix}
\]
and it has no quadratic inverse, and so on...
The simplest case of the map (65) is
\[ f(x, y, z) = \begin{pmatrix} 1 + a_3 z + a_4 x^2 \\ x \\ y \end{pmatrix} \] (66)
or equivalently
\[ f(x, y, z) = \begin{pmatrix} 1 + a_1 z + a_2 y - x^2 \\ x \\ y \end{pmatrix} \] (67)

or equivalently
\[ f(x, y, z) = \begin{pmatrix} y \\ z \\ 1 + a_1 x + a_2 y - z^2 \end{pmatrix} \] (68)
and it is studied in Ref. [7] where the attractor in this case is very similar to the Lorenz attractor with a lacuna from the Shimizu-Morioka system [1, 2], and it is shown in Fig. 5(b).

The second case is given by
\[
\begin{align*}
f(x, y, z) &= \begin{pmatrix} a_0 + a_1 x + a_2 y + a_3 z + a_5 y^2 \\ b_0 + b_1 x + b_2 y + b_3 z \\ c_0 + c_1 x + c_2 y + c_3 z \end{pmatrix} \\
c_1 b_3 - b_1 c_3 &= 0 \\
\psi_1 \left( (a_i, b_i, c_i) \bigg|_{1 \leq i \leq 9} \right) &\neq 0
\end{align*}
\] (69)

This case is also introduced in Ref. [7]. The map (70) is the inverse of a special case of the map (68), and so its attractors are repellers of map (68), and it was shown in Ref. [7] that its attractors are quite different from the attractors of map (68).

The other forms of simple 3-D quadratic diffeomorphisms are given by
\[
\begin{align*}
f(x, y, z) &= \begin{pmatrix} a_0 + a_1 x + a_2 y + a_3 z + a_7 x y \\ b_0 + b_1 x + b_2 y + b_3 z \\ c_0 + c_1 x + c_2 y + c_3 z \end{pmatrix} \\
c_1 b_3 - b_1 c_3 &= 0 \\
b_3 c_2 - b_2 c_3 &= 0 \\
\psi_1 \left( (a_i, b_i, c_i) \bigg|_{1 \leq i \leq 9} \right) &\neq 0
\end{align*}
\] (71)

The full dynamics of some simple cases of the maps
\[
\begin{align*}
f(x, y, z) &= \begin{pmatrix} a_0 + a_1 x + a_2 y + a_3 z + a_9 y z \\ b_0 + b_1 x + b_2 y + b_3 z \\ c_0 + c_1 x + c_2 y + c_3 z \end{pmatrix} \\
b_2 c_1 - b_1 c_2 &= 0 \\
c_1 b_3 - b_1 c_3 &= 0 \\
\psi_1 \left( (a_i, b_i, c_i) \bigg|_{1 \leq i \leq 9} \right) &\neq 0
\end{align*}
\] (73)
Fig. 3  Regions of dynamical behaviors in the $a_3a_7$-plane for the map (75).

Fig. 4  Regions of dynamical behaviors in the $a_1a_9$-plane for the map (76).

Fig. 5  (a) Chaotic attractor obtained from the map (66) with $a_4 = -1.6$, $a_3 = 0.1$; (b) Chaotic attractor obtained from the map (68) with $a_1 = 0.7$, $a_2 = 0.85$; (c) Chaotic attractor obtained from the map (74) with $a_1 = 0.4$, $a_2 = -0.4$; (d) Chaotic attractor obtained from the map (75) with $a_3 = -1$, $a_7 = -1.8$; (e) Chaotic attractor obtained from the map (76) with $a_1 = 0.5$, $a_9 = -0.83$. 
ified adaptive method. In these schemes the transmitted
shift key strategies, are then proposed based on the mod-
ing chaotic masking, chaotic modulation, and chaotic
control approach. Some communication schemes, includ-
tem by using the Lyapunov method and the adaptive
known parameters is discussed for a unified chaotic sys-
tics analysis, flow dynamics and liquid mixing, encryp-
tion and communications, and so on [18–20]. As a
new application of chaos theory the example given in
Ref. [20] where the adaptive synchronization with un-
known parameters is discussed for a unified chaotic sys-
tem, which effectively blur the constructed return map and can resist this return map attack. The driving
system with unknown parameters and functions is al-
most completely unknown to the attackers, so it is more
secure to apply this method into the communication.

A symplectic map is a diffeomorphism that preserves
the area, i.e. the determinant of its Jacobian matrix is
one. In the actual work, the 3-D quadratic diffeo-

morphism is a symplectic map if and only if

$$\psi_1 = 1$$

(77)

On the other hand, Poincaré sections, especially re-
turn maps provide tools for the location and stability
of resonances and periodic orbits and the existence or
nonexistence of invariant tori on the behavior of con-
tinuous time systems, such as the Lorenz system which
is the best close example connected with physics, the
Chen system, Lu system and generalized Lorenz system
[18]. It is well known that any periodic (autonomous)
Hamiltonian system of degrees of freedom generates a
2n-dimensional symplectic map by following the flow for
one period (parametrized by the value of the Hamilto-
nian, by considering the first return to a surface of section). Twist maps corresponding to Hamiltonians have
the following properties [7–11]:

(1) They are for where the velocity is a monotonic
function of the canonical momentum.

(2) They have a Lagrangian variational formulation.

(3) One-parameter families of twist maps typically ex-
hibit all possible types of dynamics, and the properties
of the minimizing orbits (the periodic and quasiperiodic
orbits) can be found throughout these transitions from
simple or integrable motion to complex or chaotic mo-
tion. The minimizing orbits are used in the theory of
transport that deals with the motion of ensembles of tra-
jectories.

In a virtual viewpoint, every model of physical phe-
nomena is a dynamical system. Almost all fundamental
models of physics are Hamiltonian dynamical systems
which gives rise to symplectic mappings. For example,
the mapping defined by a Hamiltonian flow taking an
initial condition to a state some finite time later is a
symplectic map. These mappings are included in the
study of chemical reactions, or magnetic plasma confine-
ment, especially in magnetic field line mapping, guiding
center motion, and plasma wave heating. They are also
in charged-particle motion in particle accelerators [15],
where a charged particle (either have a positive, negative
or no charge) is a particle with an electric charge. It may
be either a subatomic particle or an ion. A plasma or the

$$f(x, y, z) = \begin{pmatrix}
1 + a_1 x + a_2 y - z^2 \\
x \\
y
\end{pmatrix}$$

(74)

$$f(x, y, z) = \begin{pmatrix}
1 + a_1 x + a_0 y z \\
x \\
y
\end{pmatrix}$$

(75)

$$f(x, y, z) = \begin{pmatrix}
1 + a_3 z + a_7 x y \\
x \\
y
\end{pmatrix}$$

(76)

are shown in Figs. 2, 3, and 4, respectively. Some cor-
responding chaotic attractors are also depicted in Fig.
5(c)–(e).

Note that these cases are characterized by a typical
quasi-periodic route to chaos, contrary to the situation
for the maps given in (66) and (68) [5–7]. This confirms
that the maps (66) and (68) are not topologically equiv-
alent to the maps (74), (75), and (76).

### 3 Applications in physics

Several researchers have defined and studied quadratic
3-D chaotic systems. The first was Lorenz [17] in 1963,
where he proposed a simple three linked nonlinear differ-
ential equations with complex behavior as a model of a
weather system showing rates of change in temperature
and wind speed. The behavior of this system was sensi-
tively dependent on the initial conditions of the model,
therefore, the prediction of a future state of the system
was impossible. Using computer simulation, Ruelle in
1979 show that the first fractal shape identified took the
form of a butterfly (the butterfly effect). This justified
the usual case where weather prognostication is involved
and notoriously wrong (butterfly in the Amazon might,
in principle, ultimately alter the weather in Kansas).

Recently, chaos has been very useful in many tech-
nological disciplines such as in information and com-
puter sciences, power systems protection, biomedical sys-
tems analysis, flow dynamics and liquid mixing, encryp-
tion and communications, and so on [18–20]. As a
new application of chaos theory the example given in
Ref. [20] where the adaptive synchronization with un-
known parameters is discussed for a unified chaotic sys-
tem by using the Lyapunov method and the adaptive
control approach. Some communication schemes, includ-
ing chaotic masking, chaotic modulation, and chaotic
shift key strategies, are then proposed based on the mod-
ified adaptive method. In these schemes the transmitted
signal is masked by chaotic signal or modulated into the
system, which effectively blurs the constructed return map and can resist this return map attack. The driving
system with unknown parameters and functions is al-
most completely unknown to the attackers, so it is more
secure to apply this method into the communication.

A symplectic map is a diffeomorphism that preserves
the area, i.e. the determinant of its Jacobian matrix is
one. In the actual work, the 3-D quadratic diffeomor-
phism is a symplectic map if and only if

$$\psi_1 = 1$$

(77)
fourth state of matter is a collection of charged particles, or even a gas containing a proportion of charged particles. The simplest accelerator is the cyclotron which consists of a constant magnetic field and a time-dependent voltage drop across a narrow azimuthal gap. The motion of a fluid particle in an incompressible fluid is also Hamiltonian [16].

As an application in the electronic engineering of the 3-D quadratic diffeomorphism given by (15), the following example given in Ref. [6] where a discrete-component electronic implementation of a discrete-time hyperchaotic generalized Hénon map of the form:

\[
\begin{align*}
    x_1(k+1) &= 1.76 - x_2^2(k) - 0.1x_3(k) \\
    x_2(k+1) &= x_1(k) \\
    x_3(k+1) &= x_2(k)
\end{align*}
\]  (78)

with the initial conditions \(x_1(0) = 1, x_2(0) = 0.1, x_3(0) = 0\), the map (78) exhibits a hyperchaotic attractor.

Using analog states, the corresponding circuit designs are relatively simple and uses commonly available parts and is readily constructed. Also, experimental results show that the circuit is functional.

4 Conclusion

In this paper all possible forms of the 3-D quadratic diffeomorphisms are determined. Some numerical results are also given and discussed.

References

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