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Dynamical Models of Love

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Abstract. *Following a suggestion of Strogatz, this paper examines a sequence of dynamical models involving coupled ordinary differential equations describing the time-variation of the love or hate displayed by individuals in a romantic relationship. The models start with a linear system of two individuals and advance to love triangles and finally to include the effect of nonlinearities, which are shown to produce chaos.*

Key Words. Love, romance, differential equations, chaos.

INTRODUCTION

The power of mathematics has rarely been applied to the dynamics of romance. In his book, Strogatz (1994) has a short section on love affairs and several related mathematical exercises. Essentially the same model was described earlier by Rapoport (1960), and it has also been studied by Radzicki (1993). Related discrete dynamical models of the verbal interaction of married couples have recently been proposed by Gottman, Murray, Swanson, Tyson, & Swanson (2002). Although Strogatz's model was originally intended more to motivate students than as a serious description of love affairs, it makes several interesting and plausible predictions and suggests extensions that produce an even wider range of behavior. This paper is written in the same spirit and extends the ideas to love triangles including nonlinearities, which are shown to produce chaos.

An obvious difficulty in any model of love is defining what is meant by love and quantifying it in some meaningful way (Sternberg & Barnes 1988). There are many types of love, including intimacy, passion, and commitment (Sternberg, 1986), and each type consists of a

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complex mixture of feelings. In addition to love for another person, there is love of oneself, love of life, love of humanity, and so forth. Furthermore, the opposite of love may not be hate, since the two feelings can coexist, and one can love some things about one's partner and hate others at the same time. It is obviously unrealistic to suppose that one's love is only influenced by one's own feelings and the feelings of the other person, independent of other influences and that the parameters that characterize the interaction are unchanging, thereby excluding the possibility of learning and adaptation (Scharfe & Bartholomew, 1994). However, the goal here is to illustrate the complexity that can arise in even a minimal dynamical model when the equations are nonlinear and/or they involve three or more variables. While there is no limit to the ways in which the models can be made more realistic by adding additional phenomena and parameters, these embellishments almost certainly only increase the likelihood of chaos, which is the main new observation reported here.

SIMPLE LINEAR MODEL

Strogatz (1994) considers a love affair between Romeo and Juliet, where $R(t)$ is Romeo's love (or hate if negative) for Juliet at time t and $J(t)$ is Juliet's love for Romeo. The simplest model is linear with

$$\frac{dR}{dt} = aR + bJ$$
$$\frac{dJ}{dt} = cR + dJ \tag{1}$$

where a and b specify Romeo's "romantic style," and c and d specify Juliet's style. The parameter a describes the extent to which Romeo is encouraged by his own feelings, and b is the extent to which he is encouraged by Juliet's feelings. Gottman *et al.* (2002) use the term "behavioral inertia" for the former and "influence function" for the latter, although the inertia is greatest when $a = 0$. The resulting dynamics are two-dimensional, governed by the initial conditions and the four parameters, which may be positive or negative.

A similar linear model has been proposed by Rinaldi (1998a) in which a constant term is added to each of the derivatives in Eq. 1 to account for the appeal (or repulsion if negative) that each partner presents to the other in the absence of other feelings. Such a model is more realistic since it allows feelings to grow from a state of indifference and provides an equilibrium not characterized by complete apathy. However, it does so at the expense of introducing two additional parameters. While the existence of a non-apathetic equilibrium may be very important to the individuals involved, it does not alter the dynamics other than to move the origin of the *RJ* state space.

ROMANTIC STYLES

Romeo can exhibit one of four romantic styles depending on the signs of a and b, with names adapted from those suggested by Strogatz (1994) and his students:

1. *Eager beaver*: $a > 0$, $b > 0$ (Romeo is encouraged by his own feelings as well as Juliet's.)

2. *Narcissistic nerd:* $a > 0$, $b < 0$ (Romeo wants more of what he feels but retreats from Juliet's feelings.)

3. *Cautious (or secure) lover:* $a < 0$, $b > 0$ (Romeo retreats from his own feelings but is encouraged by Juliet's.)

4. Hermit: $a < 0$, $b < 0$ (Romeo retreats from his own feelings as well as Juliet's.)

Gragnani, Rinaldi, and Feichtinger (1997) use the terms "secure" and "synergic" to refer to individuals with negative a and positive b, respectively, and such people probably represent the majority of the population. A secure individual ($a < 0$) suppresses his feelings of love or hate in a time $-1/a$ when the other ceases to have feelings toward him, such as at death. A non-synergic individual ($b < 0$) or "nerd" is one who hates to be loved and loves to be hated. Since Juliet can also exhibit four styles, there are 16 possible pairings, each with its own dynamics, although half of those correspond to an interchange of R and J.

LOVE DYNAMICS

Equations 1 have a single equilibrium at $R = J = 0$, corresponding to mutual apathy, or a loving plateau in Rinaldi's (1988) model, with behavior determined by the eigenvalues

$$\lambda = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{(a+d)^2 - 4(ad - bc)} \qquad (2)$$

The solutions are real if $(a + d)^2 \geq 4(ad - bc)$ or a complex conjugate pair otherwise. The corresponding dynamics in the *RJ* plane are summarized in Fig. 1. The complex conjugate solution describes a focus that is stable (attracting) for $a + d$ negative and unstable (repelling) if positive, in which case all solutions are unbounded (they go to infinity). If $a + d$ is exactly zero, the solution cycles endlessly around a center at the origin. The real solutions are of two types, a node if the eigenvalues are of the same sign and a saddle otherwise. The node may either be stable (an attractor) if both eigenvalues are negative or unstable (a repellor) if both are positive. The saddle has a stable direction along which trajectories approach the origin (the inset or stable manifold) and an unstable direction along which they are repelled (the outset or unstable manifold). Strogatz (1994) asks his students to consider a number of special pairings of individuals as described in the following sections.

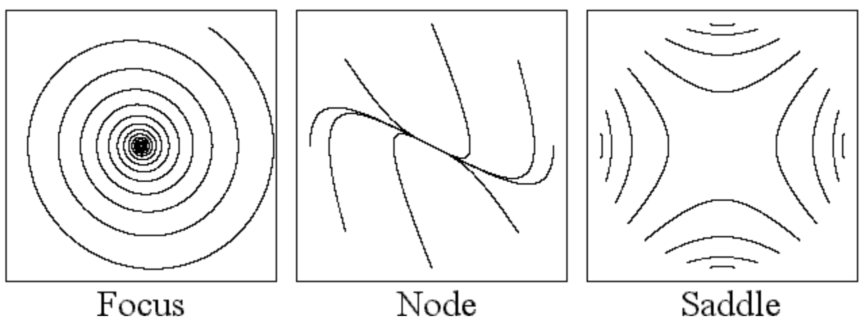

Fig. 1. Dynamics in the vicinity of an equilibrium point in two dimensions from Eq. 1.

OUT OF TOUCH WITH ONE'S OWN FEELINGS

Consider the special case with both Romeo and Juliet out of touch with their own feelings ($a = d = 0$) and only responding to the other. The eigenvalues are $\lambda = \pm\sqrt{bc}$, and the dynamics then depend on b and c, for which there are three combinations with the outcomes indicated:

1. *Two lovers:* $b > 0$, $c > 0$ (Saddle, mutual love or mutual hate).
2. *Two nerds:* $b < 0$, $c < 0$ (Saddle, one loving and the other hating).
3. *Nerd plus lover:* $bc < 0$ (Center, endless cycle of love and hate).

The outcome for Cases 1 and 2 depend on the initial conditions (first impressions count) as does the size of the oscillation in Case 3.

FIRE AND ICE

Now consider the case where the two lovers are exact opposites ($c = -b$ and $d = -a$). The eigenvalues are $\lambda = \pm\sqrt{a^2 - b^2}$, and the dynamics then depend on a and b, for which there are two combinations:

1. *Eager beaver plus hermit:* $ab > 0$.
2. *Narcissistic nerd plus cautious lover:* $ab < 0$.

The outcome depends on whether the individuals respond more to themselves ($|a| > |b|$) or to the other ($|a| < |b|$). The former case leads to a saddle in which the eager beaver and hermit are at odds and the narcissistic nerd and cautious lover are in love or at war, and the latter leads to a center. Thus they can end up in any quadrant (all four combinations of love and hate) or in a never-ending cycle, but never apathetic.

PEAS IN A POD

Two romantic clones ($c = b$ and $d = a$) have eigenvalues $\lambda = a \pm b$ and dynamics that depend on a and b. Cautious lovers with $|a| < |b|$ and eager beavers end up in either a love fest or war depending on the initial conditions. Hermits with $|a| < |b|$ and narcissistic nerds end up with one loving and the other hating. Cautious lovers and hermits with $|a| > |b|$ end up in a state of mutual apathy. Oscillations are not possible.

ROMEO THE ROBOT

Suppose Romeo's feelings toward Juliet are unaffected by her feelings for him ($b = 0$) as well as his feelings toward her ($a = 0$), so that R is a constant and the eigenvalues are $\lambda = d$ and $\lambda = 0$. Then there is an equilibrium in which Juliet's feelings are given by $J = -cR/d$, which may be positive or negative depending on the sign of R and her romantic style. If Romeo loves Juliet ($R > 0$), she will love him back only if she is a cautious lover or a narcissistic nerd ($cd < 0$). However, the equilibrium is stable only if she is cautious ($d < 0$). If she is narcissistic ($d > 0$), either her love will grow without bound or she will come to hate him depending on the initial conditions, but her feelings will never die and oscillations are not possible.

LOVE TRIANGLES

Richer dynamics can occur if a third party is added, in part because alliances can arise in which two members can ally against the third. Suppose Romeo has a mistress, Guinevere, although the third person could be a child or other relative. The state space is then six-dimensional rather than two-dimensional because each person has feelings for two others and there are twelve parameters if each can adopt different styles toward the others, even when the natural appeal considered by Rinaldi (1998a) is ignored.

In the simplest case, Juliet and Guinevere would not know about one another and Romeo would adopt the same romantic style toward them both (he is an equal opportunity lover). The resulting four-dimensional system becomes two decoupled two-dimensional systems unless Romeo's feelings for Juliet are somehow affected by Guinevere's feelings for him, and similarly for Guinevere. For simplicity, suppose Guinevere's feelings for Romeo affect his feelings for Juliet in a way that is exactly opposite to the way Juliet's feelings for him affect his feelings for her, and similarly for Juliet. The corresponding model is then:

$$\begin{aligned}\frac{dR_J}{dt} &= aR_J + b(J - G)\\ \frac{dJ}{dt} &= cR_J + dJ\\ \frac{dR_G}{dt} &= aR_G + b(G - J)\\ \frac{dG}{dt} &= eR_G + fG\end{aligned} \qquad (3)$$

Note that in this simple model, his total love $R_J + R_G$ either increases without bound (if $a > 0$) or dies (if $a < 0$), independent of the other parameters, although he may still have very different feelings toward his lovers in the process ($R_J \neq R_G$), depending on their characteristics. This curious result arises because he reacts to them identically so that their effects on him exactly cancel. In some ways he treats them as a single individual with characteristics given by their average. This situation is similar to the case with two persons whose love also dies or grows without bound except in the special case of $a + d = 0$ and $ad > bc$. The symmetry can be easily broken, but at the expense of adding additional parameters to the model.

Solutions resembling Fig. 1 are common, but other behaviors are also possible, such as in Fig. 2. The trajectory in R_JJ space appears to intersect itself, but that is because this plane is a projection of a four-dimensional dynamic. A mathematically astute Juliet would infer the additional variables. All solutions either attract to the origin or go to infinity. This particular case corresponds to an eager beaver Romeo (a =

b = 1) whose lovers are both narcissistic nerds ($c = e = -1$ and $d = f = 1$). The trajectory and outcome depend on the initial conditions, but for this case, Romeo eventually loves Juliet and hates Guinevere, and they both hate him.

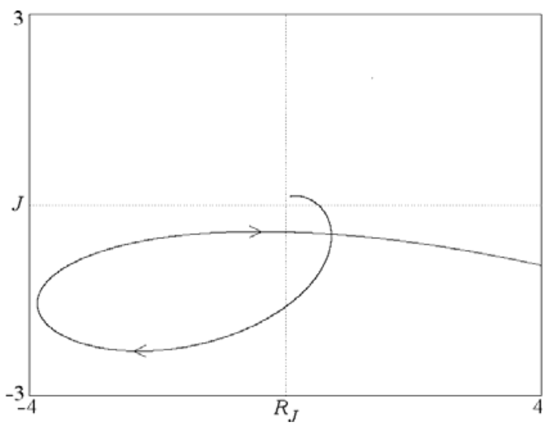

Fig. 2. One solution of the linear love triangle in Eq. 3.

It is interesting to ask how Romeo fares when averaged over all romantic styles and initial conditions. Taking the parameters a through f and the initial conditions from a random Gaussian distribution with mean zero and variance one shows that Romeo ends up hating one and loving the other 82% of the time and loving or hating both 8% of the time. In 10% of the cases everyone is apathetic, and no one can be apathetic unless everyone is. In only 2% of the cases is everyone in love, and similarly for hate. Thus the prognosis for this arrangement is rather dim, although someone is likely to enjoy it by experiencing mutual love in 43% of the cases (at least until she discovers her competition!).

NONLINEAR EFFECTS

The foregoing discussion involved only linear equations for which the allowable dynamics are limited. There are countless ways to introduce nonlinearities. Imagine that Romeo responds positively to Juliet's love, but if she loves him too much, he feels smothered and reacts adversely. Conversely, if Juliet is sufficiently hostile, Romeo might decide to be nice to her, in what Gottman *et al.* (2002) call the "repair nonlinearity." Thus we could replace the bJ in Eq. 1 with the logistic function $bJ(1 - |J|)$, which amounts to measuring J in units such that $J = 1$ corresponds to the value at which her love become counterproductive. Qualitatively similar results follow from the function $bJ(1 - J^2)$, which is the case considered by Rinaldi (1998b) in his model of the love felt by the 14th century Italian poet Francis Petrarch (1304-1374) toward his beautiful married platonic mistress Laura (Jones 1995). Assuming the same for Juliet gives:

$$\begin{aligned}\frac{dR}{dt} &= aR + bJ(1-|J|)\\ \frac{dJ}{dt} &= cR(1-|R|) + dJ\end{aligned} \tag{4}$$

There are now four equilibria, including the one at the origin. Figure 3 shows a stable focus in which an eager beaver Juliet ($c = d = 1$) leads a hermit Romeo ($a = b = -2$) to a mutually loving state with $R = J = 2$. A similar model for cautious (secure) lovers with a sigmoid nonlinearity also gives stable equilibria (Rinaldi & Gragnani 1998). Equations 4 apparently do not admit limit cycles, and there is no chaos since the system is only two-dimensional.

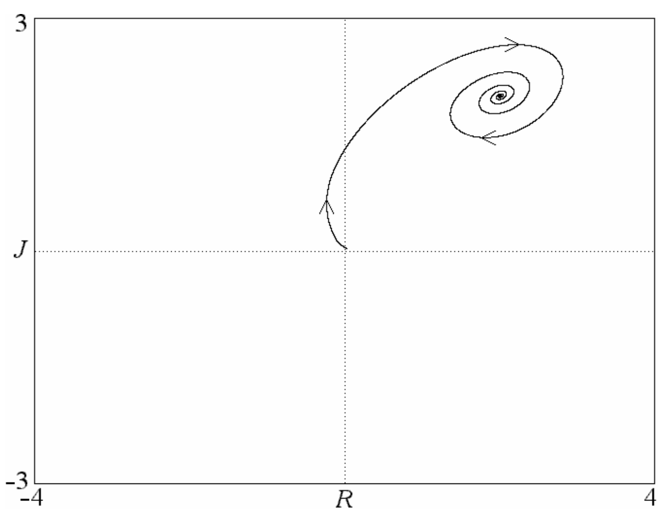

Fig. 3. One solution of the nonlinear Romeo-Juliet dynamic in Eq. 4.

The same nonlinearity can be applied to the love triangle in Eq. 3 giving

$$
\begin{aligned}
\frac{dR_J}{dt} &= aR_J + b(J-G)(1-|J-G|)\\
\frac{dJ}{dt} &= cR_J(1-|R_J|) + dJ\\
\frac{dR_G}{dt} &= aR_G + b(G-J)(1-|G-J|)\\
\frac{dG}{dt} &= eR_G(1-|R_G|) + fG
\end{aligned}
\qquad (5)
$$

This system can exhibit chaos with strange attractors, an example of which is in Fig. 4. This case has a cautious lover Romeo (a = –3 and b = 4) and Guinevere (e = 2 and f = –1) and a narcissistic nerd Juliet (c = –7 and d = 2). The largest Lyapunov exponents (base-e) are 0.380, 0, and –14.380, and the rate of contraction in state space (the sum of the Lyapunov exponents) is $2a + c + f = –14$. The Kaplan-Yorke dimension is 2.026 (Sprott 2003). Figure 5 illustrates the effect of the positive Lyapunov exponent on the time evolution of Romeo's love for Juliet for the same case as Fig. 4 but with two initial conditions that are identical except that Romeo's love for Juliet differs by 1%. The regions of parameter space that admit chaos are relatively small, sandwiched between cases that produce limit cycles and unbounded solutions.

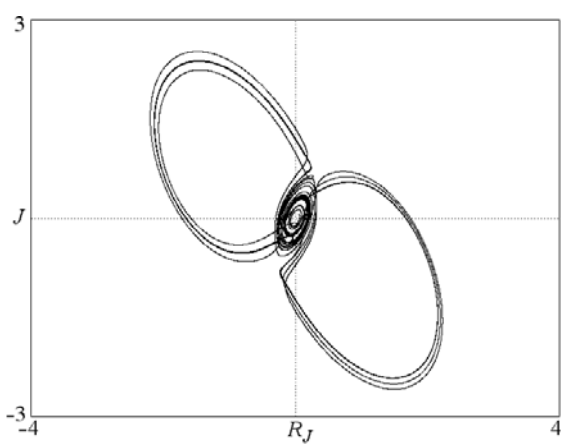

Fig. 4. Strange attractor from the nonlinear love triangle in Eq. 5.

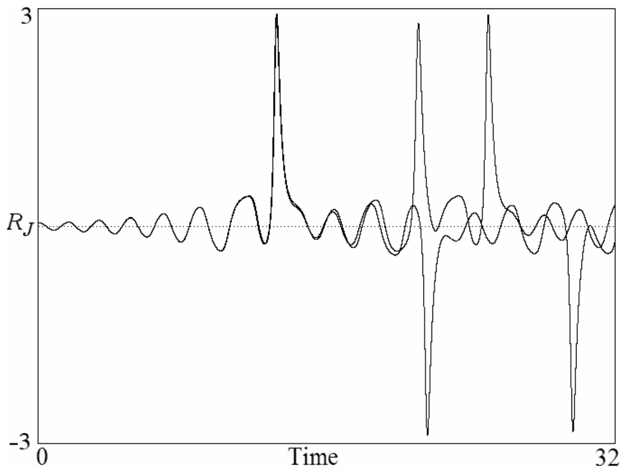

Fig. 5. Chaotic evolution of Romeo's love for Juliet from Eq. 5 showing the effect of changing the initial conditions by 1%.

CONCLUSIONS

Simple linear models of love can produce surprisingly complex dynamics, much of which rings true to common experience. Even simple nonlinearities can produce chaos when there are three or more variables. An interesting extension of the model would consider a group of interacting individuals (a large family or love commune). The models are gross simplifications since they assume that love is a simple scalar variable and that individuals respond in a consistent and mechanical way to their own love and to the love of others toward them without external influences.

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