Chaos and the Limits of Predictability for the Solar-Wind Driven Magnetosphere-Ionosphere System

W. Horton[1], R.S. Weigel[1], and J. Clint Sprott[2]

[1]Institute for Fusion Studies, The University of Texas, Austin, TX 78712, U.S.A.
[2]Department of Physics, University of Wisconsin, Madison, WI 53706, USA

PPD-APS Quebec, Canada

October 23-27, 2000

The Substorm Problem

- The solar wind driven magnetosphere-ionosphere is a complex driven-damped dynamical system.

- There is a great variety of observational forms in the substorm category of MI events suggesting that the appropriate behavior is one of a chaotic system.

- There is global spatial coherence as emphasized by Baker *et al.* (1999) in the correlated measurements from

  (1) the ground-based auroral latitude chain of magnetometers whose wave forms are used to give the classic definition of a substorm event by the AL geomagnetic wave form.

  (2) the satellite measurements of particle distributions and electromagnetic fields in the geotail plasma containing the large cross-tail current loop $I(t)$ that provides confinement of the high mean plasma pressure $p(t)$ plasma.

The WINDMI Substorm Model

Here we investigate two questions concerning the WINDMI description of the substorm dynamics.

Questions:

Q1: We ask, can the $d = 6$ energy component system be reduced to the minimal order $d = 3$ system of a deterministic chaotic system described by ordinary differential equations?

In particular what is lost by the series of reductions presented here?

- In making the reduction to $d = 3$ the faster evolving energy components are taken analytically to be given by these local fixed point or "equilibrium" values. In making this replacement, the energy and charge conserving features of the full ode system are lost during the short-lived transients.
- The only practical method of judging how much fidelity is lost by these reductions is by comparing the dynamics of the full and reduced system.

Q2: Secondly, we ask what does the minimal model tell us about the intrinsic limits of predictability in the system?

Lyapunov Exponents and Lyapunov Fractal Dimension

- The largest Lyapunov exponent, designated as LE, determines the rate of divergence of neighboring trajectories. For a positive LE the reciprocal is the time for one e-folding in the distance between neighboring orbits.

  This divergence time sets the limit of predictability. For typical solar wind conditions we will show that the intrinsic time limit is of order three hours ($L_E^{-1} \sim 3\,\mathrm{hr}$) for moderately strong solar wind driving voltages.

- Determining all three Lyapunov exponents for the minimal model also allows us to determine the Lyapunov dimension (Kaplan and York, 1979) of the chaotic attractor for the system of $D_L \cong 2.1$. We compare this finding from a level of 20 kV ($V_{\mathrm{sw}} = 1$) to a maximum of approximately $v = 200\,\mathrm{kV}$ ($V_{\mathrm{sw}} = 10$) with that given earlier for a very simple analog model made up of a Rössler system driving a Lorenz system in Horton *et al.* (1999) and Doxas *et al.*

The WINDMI Substorm Model Equations

The WINDMI dynamical model of the Earth's magnetosphere is a set of six first-order ordinary differential equations whose solutions are chaotic for values of the parameter vector **P** of interest. Smith *et al.* (2000) expressed these equations in dimensionless form as

$$\frac{dI}{dt} = a_1(V_{\text{sw}} - V) + a_2(V - V_i) \tag{1}$$

$$\frac{dV}{dt} = b_1(I - I_1) - b_2 P^{1/2} - b_3 V \tag{2}$$

$$\frac{dP}{dt} = V^2 - K_{\|}^{1/2} P\{1 + \tanh[d_1(I - 1)]\}/2 \tag{3}$$

$$\frac{dK_{\|}}{dt} = P^{1/2} V - K_{\|} \tag{4}$$

$$\frac{dI_1}{dt} = a_2(V_{\text{sw}} - V) + f_1(V - V_i) \tag{5}$$

$$\frac{dV_i}{dt} = g_1 I_1 - g_2 V_i - g_3 I_1^{1/2} V_i^{3/2} \tag{6}$$

where the coupled current loop equations $(\dot{I}, \dot{I}_1)$ have been diagonalized. The ten dimensionless parameters define a parameter vector **P** that characterizes the global state of the magnetosphere-ionosphere system. A system that appears to conform closely to the known wave forms of the classic internally-triggered type I substorms is defined by the following parameter values: $\mathbf{P}(S3) = [a_1 = 0.247, a_2 = 0.391, b_1 = 10.8, b_2 = 0.0752, b_3 = 1.06, d_1 = 2200, f_1 = 2.47, g_1 = 1080, g_2 = 4, g_3 = 3.79]$. General properties of four states $S1, S2, S3, S4$ are discussed in Smith *et al.* (2000).

The bifurcation sequence for the $S3$ state as the solar wind voltage increases from $V_{\text{sw}} = 0$ to 10 ($\simeq 200\,\text{kV}$) is shown in Fig. 1.

Reduction to a Minimal 3D Dynamical Model

A reduced form of these equations results from setting the last two time derivatives to zero and solving for I_1 and V_i, giving

$$V_i = V + a_2(V_{\rm sw} - V)/f_1 \tag{7}$$

$$I_1 = g_3^2 V_i^3/2g_1^2 + g_2 V_i/g_1 + \left(g_3 V_i^2/2g_1^2\right)\left(4g_1 g_2 + g_3^2 V_i^2\right)^{1/2}. \tag{8}$$

The full solution relaxes to these fixed points in the (nonlinear) R_1C_1-time scale of $1/[g_2+g_3(I_1V_1)^{1/2}]$ which for the $S3$ state is short $\lesssim 10^{-1}$. Substituting Eqs. (7) and (8) into Eqs. (1)-(4) yields the first reduced system.

The reduced 4-dimension system was solved numerically, and the results are very similar to the full 6-dimensional case, shown in Fig. 1. Figure 2 shows the largest Lyapunov exponent (base-e) as a function of $V_{\rm sw}$.

From numerical experiments, it turns out that a_2, g_2, and g_3 can all be set to zero simultaneously without destroying the chaos and gives a plot very similar to Fig. 2. Furthermore, Eq. (4) can apparently be eliminated as well, giving $K_{||} = P^{1/2}V$, for which chaos still occurs. Finally, the cross-tail voltage V in Eq. (3) can be replaced with its equilibrium value of $V_{\rm sw}$. Putting in all these simplifications gives the following 3-D system:

$$\frac{dI}{dt} = a_1(V_{\rm sw} - V) \tag{9}$$

$$\frac{dV}{dt} = b_1 I - b_2 P^{1/2} - b_3 V \tag{10}$$

$$\frac{dP}{dt} = V_{\rm sw}^2 - P^{5/4} V_{\rm sw}^{1/2}\{1 + \tanh[d_1(I-1)]\}/2. \tag{11}$$

Dimensionless Reduced System

Now define new variables,

$$x = d_1(I - 1) \tag{12}$$

$$y = a_1 d_1 (V - V_{\text{sw}}) \tag{13}$$

$$z = a_1 b_2 d_1 P^{1/2} + a_1 d_1 (b_3 V_{\text{sw}} - b_1) \tag{14}$$

in terms of which the 3-D system above can be written as

$$\frac{dx}{dt} = -y \tag{15}$$

$$\frac{dy}{dt} = c_1 x - b_3 y - z \tag{16}$$

$$\frac{dz}{dt} = -c_2 - c_3 \tanh(x) \tag{17}$$

where

$$c_1 = a_1 b_1,$$

$$c_2 = \frac{1}{4} a_1 d_1 (b_1 - b_3 V_{\text{sw}})^{3/2} (V_{\text{sw}}/b_2)^{1/2} - a_1 b_2^2 d_1 V_{\text{sw}}^2 / 2(b_1 - b_3 V_{\text{sw}}),$$

and

$$c_3 = \frac{1}{4} a_1 d_1 (b_1 - b_3 V_{\text{sw}})^{3/2} (V_{\text{sw}}/b_2)^{1/2}.$$

The condition that $b_3 V_{\text{sw}} < b_1$ is that there be a finite part of the cross-tail current driven by the MHD pressure gradient $j \times B = \nabla p$ condition at the point where I hits the critical current I_c.

Properties of the Minimal Model

The system has thus been reduced to one with 6 terms, 4 parameters, and a single nonlinearity. To verify that the approximations are reasonable, the largest Lyapunov exponent for this system is plotted versus $V_{\rm sw}$ in Fig. 3a. Figure 3b shows the Lyapunov dimension D_L and Fig. 3c the bifurcation diagram for the reduced system. The dimension in the chaotic regime is only slightly greater than 2.0 and is consistent with calculations of the correlation dimension (not shown).

From the fidelity of the reduced dynamics, we have answered Q1 and shown that the hyperbolic tangent is the important nonlinearity producing the chaos. The 3-D system has a fixed point at $x^* = -\tanh^{-1}(c_2/c_3)$, $y^* = 0$, $z^* = c_1 x^*$ with eigenvalues λ that satisfy

$$\lambda^3 + b_3\lambda^2 + c_1\lambda + c_3 - c_2^2/c_3 = 0.$$

A Hopf bifurcation occurs at $c_1 b_3 = c_3 - c_2^2/c_3$ followed by a period doubling route to chaos.

Lyapunov Exponential Fractal Dimension

For this system, the sum of the Lyapunov exponents is $-b_3$. From the calculated value for the largest exponent and the fact that one exponent must be zero, the entire spectrum can be obtained for the above parameters with $V_{\rm sw} = 4.8$ as $(0.144, 0, -1.204)$. (The value $V_{\rm sw} = 4.91$ used for Fig. 1 is in a periodic window.) The Lyapunov dimension is $2 + 0.144/1.204 = 2.12$ in rough agreement with the calculated correlation dimension of 2.13.

“Jerk” Systems

The simplification can be carried one step further by reducing the system to a jerk form (Sprott, 2000a):

$$\frac{d^3x}{dt^3} + b_3 \frac{d^2x}{dt^2} + c_1 \frac{dx}{dt} = -c_2 - c_3 \tanh(x) \tag{18}$$

for a third-order explicit ODE with a scalar variable. This is a special case of a damped harmonic oscillator driven by a nonlinear memory term, whose solutions have been studied and are known to be chaotic. The period of the dominant frequency of oscillation is $2\pi/c_1^{1/2} = 3.85$, corresponding to a real time of about 1 hour. In Eq. (18), t can be rescaled by $c_1^{1/2}$ to eliminate one of the four coefficients. The dynamical behavior is thus determined by only three parameters $\mathbf{P}_3 = (b_3, C_2, C_3)$. The regions of various dynamical behaviors are plotted in the $b_3 - V_{\mathrm{sw}}$ plane in Fig. 4 with the other parameters as listed above.

Analog Simulations of Substorms

This system could be implemented electronically since a saturating operational amplifier produces a good approximation to the tanh(x) function, although the behavior is somewhat sensitive to the exact nature of the nonlinearity. A circuit for doing this is shown in Fig. 5. More information on the use of circuits with operational amplifiers and diodes to represent this class of physical systems is given in Sprott (2000b).

References

[1] Baker, D.N., T.I. Pulkkinen, J. Büchner, and A.J. Klimas, Substorms: A global instability of the magnetosphere–ionosphere system, J. Geophys. Res. **104**, 14,601–14,611, (1999).

[2] Horton, W. and I. Doxas, A low dimensional energy conserving state space model for substorm dynamics, J. Geophys. Res. **101A**, 27223 (1996).

Horton, W. and I. Doxas, A low dimensional dynamical model for the solar wind driven geotail–ionosphere system, J. Geophys. Res. **103A**, 4561, (1998).

[3] Horton, W., J. Smith, B. Weigel, B. Goode, I. Doxas, and J. Cary, The solar-wind driven magnetosphere-ionosphere as a complex dynamical system, Phys. Plasmas **6**(11), 4178-4184 (1999).

[4] Smith, J.P., J.-L. Thiffeault, W. Horton, Dynamical range of the WINDMI model: An exploration of possible magnetospheric plasma states, J. Geophys. Res. **105**, 12983 (2000).

[5] J. C. Sprott, A new class of chaotic circuit, Phys. Lett. A **266**, 19-23 (2000a).

[6] J.C. Sprott, Simple chaotic systems and circuits, Am. J. Phys. **68**(8), 758-763 (2000b).

[7] J. Kaplan and J. York, “ Lecture Notes in Mathematics 730,” in *Functional Differential Equations and the Approximation of Fixed Points*, (Spring Verlag, Berlin, 1979), p. 204.