A unified piecewise smooth chaotic mapping that contains the Hénon and the Lozi systems

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Abstract

In this paper we introduce a new piecewise smooth mapping of the plane as a unified discrete-time chaotic system that contains the original Hénon and Lozi systems as two extremes and other systems as a transition in between and that has robust homoclinic chaos over a portion of its key system parameters. Dynamical behaviors of the unified system are investigated in some detail.

Keywords: A unified piecewise smooth map, transition Hénon-like and Lozi-like chaotic attractors, robust homoclinic chaos.

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1 Introduction

Discrete mathematical models arise directly from experiment or by the use of the Poincaré section for the study of continuous models. Two of these models are the Hénon [1] and the Lozi [2] maps given by:

\[
H(x, y) = \left(1 - \frac{a x^2 + y}{b x}\right) \quad \text{and} \quad L(x, y) = \left(1 - \frac{a|x| + y}{b x}\right). \quad (1)
\]
The $H$ mapping gives a chaotic attractor called the Hénon attractor, which is obtained for $a = 1.4$ and $b = 0.3$ as shown in Fig. 1(a). There are many papers that discuss the original Hénon and Lozi maps such as [3-6]. Moreover, it is possible to change the form of the Hénon mapping $H$ to obtain other chaotic attractors [2, 7, 8]. Applications of these maps include secure communications using the notions of chaos [11, 12]. The Lozi map $L$ is a 2-D non-invertible iterated map that gives a chaotic attractor called the Lozi attractor, which is obtained for $a = 1.4$ and $b = 0.3$ as shown in Fig. 1(b). It is therefore interesting to ask if there is a chaotic system that can unify these two chaotic systems and realize the continued transition from one to the other. This paper provides a positive answer to this question and reveals a surprising property of the transitional systems.

Robust chaos is defined by the absence of periodic windows and coexisting attractors in some neighborhood of the parameter space, since the existence of these windows in some chaotic regions implies that small changes of the parameters would destroy the chaos. This effect implies the fragility of this type of chaos. Contrary to this situation, there are many practical applications such as in communications and spreading the spectrum of switch-mode power supplies to avoid electromagnetic interference [13, 14] where it is necessary to obtain reliable operation in the chaotic mode and thus where robust chaos is required. A practical example can be found from electrical engineering to demonstrate robust chaos as shown in [10]. The occurrence of robust chaos in a smooth system is proved and discussed in [16] along with a general theorem and a practical procedure for constructing S-unimodal maps that generate robust chaos. This result is contrary to the hypothesis that robust chaos cannot exist in smooth systems [10]. On the other hand, many methods are used to search for a smooth and robust chaotic map, such as in [15], where a one-dimensional smooth map that generates robust chaos in a large domain of the parameter space is presented. In [17], simple polynomial unimodal maps that show robust chaos are constructed. Other methods are given in [16-18].

Since practical applications of chaos require the chaotic orbit to be robust, we introduce in this paper a new unified chaotic system that reduces to the original Hénon and Lozi systems [1-2] as two extremes and to other systems as a transition in between, and which has robust homoclinic chaos over a portion of its key system parameters. The proposed unified chaotic model is a piecewise smooth map of the plane defined by:
Figure 1: (a) The original Hénon chaotic attractor obtained from the $H$ mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$. (b) The original Lozi chaotic attractor obtained from the $L$ mapping with its basin of attraction (white) for $a = 1.4$ and $b = 0.3$.

$$U(x, y) = \begin{pmatrix} 1 - 1.4f_\alpha(x) + y \\ 0.3x \end{pmatrix},$$

(2)

where $0 \leq \alpha \leq 1$ is the bifurcation parameter and the function $f_\alpha$ is given by:

$$f_\alpha(x) = \alpha |x| + (1 - \alpha) x^2.$$  

(3)

It is easy to remark that for $\alpha = 0$, one has the original Hénon map, and for $\alpha = 1$, one has the original Lozi map, and for $0 < \alpha < 1$, the unified chaotic map (2) is chaotic with different kinds of attractors. The Lyapunov exponents and bifurcation diagram are shown in Fig. 2. We remark that the unified chaotic map (2) is a piecewise smooth map and it is a diffeomorphism since the determinant of its Jacobian in (15) and (16) below is $-0.3$ for all $0 \leq \alpha \leq 1$. On the other hand, and due to the shape of the vector field $U$ of the unified chaotic map (2), the plane can be divided into two regions denoted by:

3
Let us define:

\[ A = \{(x, y) \in \mathbb{R}^2 / x = 0\} , \]  

which denotes a smooth curve that divides the phase plane into two regions \( D_1 \) and \( D_2 \), so that the unified chaotic map (2) can be rewritten as follow:

\[
U(x, y) = \begin{cases} 
1.4 (\alpha - 1) x^2 + 1.4 \alpha x + y + 1, & \text{if } (x, y) \in D_1 \\
1.4 (\alpha - 1) x^2 - 1.4 \alpha x + y + 1, & \text{if } (x, y) \in D_2 \\
0.3x &
\end{cases}
\]

where in each of these regions the system (2) is a quadratic map. Notably, the unified system (2) has some special features and advantages as follows:

1. System (2) is chaotic when \( 0 \leq \alpha \leq 1 \).
2. System (2) connects the Hénon and the Lozi maps and realizes the entire transition spectrum from one to the other.
3. The control parameter \( \alpha \) in system (2) reveals the evolution of dynamical behaviors from the Hénon to the Lozi attractors.
4. System (2) has robust chaotic attractors for \( 0.493122734 \leq \alpha < 1 \), while it is absent for \( \alpha = 0 \) and \( \alpha = 1 \).

2 Numerical simulations

In this section, the dynamical behaviors of the unified chaotic system (2) will be investigated numerically. For \( 0 \leq \alpha \leq 1 \), the unified chaotic system has two kinds of chaotic orbits: Hénon-like chaotic attractors over the first portion of the interval \([0, 1]\] and a Lozi-like chaotic attractor over the second portion of the interval \([0, 1]\] as shown in Fig. 3(a) and (c). It seems that this phenomenon is related to the shape of the function \( f_\alpha \), where for values of \( \alpha \) close to zero, the function \( f_\alpha \) given in (3) behaves like the quadratic term \( x^2 \), while the values of \( \alpha \) close to unity the function \( f_\alpha \) behaves like the absolute value function \(|x|\), as shown in Fig. 3(b) and (d). This explains the occurrence of the two kinds of chaotic attractors mentioned above.
Figure 2: (a) Variation of the Lyapunov exponents of the unified map (2) for $0 \leq \alpha \leq 1$. (b) The bifurcation diagram of the unified chaotic map (2) for $0 \leq \alpha \leq 1$. 


Figure 3: (a) The transition Hénon-like chaotic attractor obtained for the unified chaotic map (2) with its basin of attraction (white) for $\alpha = 0.2$. (b) The graph of the function $f_{0.2}$. (c) The transition Lozi-like chaotic attractor obtained for the unified chaotic map (2) with its basin of attraction (white) for $\alpha = 0.8$. (d) The graph of the function $f_{0.8}$.
3 A rigorous proof of the robustness of the homoclinic chaos

In this section, we begin by studying the existence of the fixed point of the $U$ mapping in order to determine the associated normal form for the unified chaotic map (2), which permits us to prove rigorously the occurrence of robust homoclinic chaos, where we exclude the values $\alpha = 0$ and $\alpha = 1$ since both the Hénon and Lozi mapping are studied in detail in several works and in the references therein. We will show that if $0 \leq \alpha < 1$, then the unified chaotic map (2) has two fixed points given by:

$$P_1 = (x_1, 0.3x_1) \in D_1 \text{ and } P_2 = (x_2, 0.3x_2) \in D_2, \quad (8)$$

where

$$\begin{cases} x_1 = \frac{-0.7\alpha + 0.35 + \sqrt{-7.56\alpha + 1.96\alpha^2 + 6.09}}{1.4(\alpha - 1)} \\ x_2 = \frac{0.7\alpha + 0.35 - \sqrt{-3.64\alpha + 1.96\alpha^2 + 6.09}}{1.4(\alpha - 1)} \end{cases} \quad (9)$$

Obviously, the fixed points of the unified chaotic map (2) are the real solutions of the system:

$$1 - 1.4f_\alpha (x) + y = x \text{ and } y = 0.3x. \quad (10)$$

Hence one may easily obtain the two equations:

$$1.4(\alpha - 1)x^2 + (1.4\alpha - 0.7)x + 1 = 0 \text{ for } x < 0 \text{ and } y = 0.3x \quad (11)$$

$$1.4(\alpha - 1)x^2 - (1.4\alpha + 0.7)x + 1 = 0 \text{ for } x > 0 \text{ and } y = 0.3x. \quad (12)$$

If $0 \leq \alpha < 1$, then $1.4(\alpha - 1) < 0$, and the descriminant of the first equation of (11) is $-7.56\alpha + 1.96\alpha^2 + 6.09 > 0$. Thus, one can easily conclude that the only negative solution of the first equation of (11) is:

$$x_1 = \frac{-0.7\alpha + 0.35 + \sqrt{-7.56\alpha + 1.96\alpha^2 + 6.09}}{1.4(\alpha - 1)} < 0. \quad (13)$$
On the other hand, the discriminant of the first equation of (12) is
\(-3.64\alpha + 1.96\alpha^2 + 6.09 > 0\) for all \(0 \leq \alpha < 1\). Thus, one can easily conclude that the only positive solution of the first equation of (12) is:

\[
x_2 = \frac{0.7\alpha + 0.35 - \sqrt{-3.64\alpha + 1.96\alpha^2 + 6.09}}{1.4(\alpha - 1)} > 0.
\] (14)

Finally, the unified chaotic map (2) has two simultaneous fixed points defined for \(0 < \alpha < 1\) as \(P_1 = (x_1, 0.3x_1) \in D_1\) and \(P_2 = (x_2, 0.3x_2) \in D_2\).

The Jacobian matrix of the unified chaotic map (2) evaluated at a point \((x, y)\) in the region \(D_1\) is given by:

\[
J_1(x, y) = \begin{pmatrix}
1.4\alpha - 2.8x + 2.8x\alpha & 1 \\
0.3 & 0
\end{pmatrix},
\] (15)

and at a point \((x, y)\) in the region \(D_2\) the Jacobian matrix is given by:

\[
J_2(x, y) = \begin{pmatrix}
2.8x\alpha - 1.4\alpha - 2.8x & 1 \\
0.3 & 0
\end{pmatrix}.
\] (16)

Thus, at \(P_1\) one has:

\[
J_1(P_1) = \begin{pmatrix}
0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} & 1 \\
0.3 & 0
\end{pmatrix}.
\] (17)

The eigenvalues of \(J_1(P_1)\) are

\[
\begin{cases}
\lambda_1 = \frac{\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} + \sqrt{1.96\alpha^2 - 7.56\alpha + 1.4\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} + 7.78}}{2} + 0.35, \\
\lambda_2 = \frac{\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} - \sqrt{1.96\alpha^2 - 7.56\alpha + 1.4\sqrt{1.96\alpha^2 - 7.56\alpha + 6.09} + 7.78}}{2} + 0.35,
\end{cases}
\] (18)

and at \(P_2\) one has:

\[
J_2(P_2) = \begin{pmatrix}
0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} & 1 \\
0.3 & 0
\end{pmatrix}.
\] (19)

The eigenvalues of \(J_2(P_2)\) are:

\[
\begin{cases}
\omega_1 = \frac{-\sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} + \sqrt{1.96\alpha^2 - 3.64\alpha - 1.4\sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} + 7.78}}{2} + 0.35, \\
\omega_2 = \frac{-\sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} - \sqrt{1.96\alpha^2 - 3.64\alpha - 1.4\sqrt{1.96\alpha^2 - 3.64\alpha + 6.09} + 7.78}}{2} + 0.35,
\end{cases}
\] (20)
In the case of two-dimensional piecewise smooth maps, it is possible to choose an appropriate coordinate transformation so that the choice of axis is independent of the parameter. In so doing, the normal form of map (1) is given by [9]:

\[
N(x, y) = \begin{cases} 
(\tau_1 \ 1) (x) + (0 \ 1) \mu, & \text{if } x < 0, \\
(\tau_2 \ 1) (x) + (0 \ 1) \mu, & \text{if } x > 0, 
\end{cases}
\]  

(21)

where \(\mu\) is a parameter, and \(\tau_i, \delta_i, i = 1, 2\) are the traces and determinants of the corresponding matrices of the linearized map in the two subregion \(D_1\) and \(D_2\) evaluated at \(P_1\) and \(P_2\) respectively, and they are given by:

\[
\begin{align*}
\tau_1 &= 0.7 + \sqrt{1.96\alpha^2 - 7.56\alpha + 6.09}, \\
\tau_2 &= 0.7 - \sqrt{1.96\alpha^2 - 3.64\alpha + 6.09}, \\
\delta_1 &= \delta_2 = -0.3,
\end{align*}
\]  

(22)

It is shown in [10] that a robust homoclinic chaos (i.e. the existence of an infinity of homoclinic intersections between the two subregions \(D_1\) and \(D_2\) ) occurs in the piecewise smooth map of the form (21) when:

\[
\begin{align*}
\tau_1 &> 1 + \delta_1, \text{ and } \tau_2 < -(1 + \delta_2), \\
\delta_1 &< 0, \text{ and } -1 < \delta_2 < 0,
\end{align*}
\]  

(23)

and the condition:

\[
\frac{\lambda_1 - 1}{\tau_1 - 1 - \delta_1} > \frac{\omega_2 - 1}{\tau_2 - 1 - \delta_2},
\]  

(24)

where the parameter range for boundary crisis is given by:

\[
(\lambda_2 - \tau_2) \lambda_1 - \tau_1 + \tau_2 + \delta_1 > 0,
\]  

(25)

because \(\delta_1 = \delta_2\), where the inequality (25) determine the condition of stability of the chaotic attractor. However, if the first condition (24) is not satisfied, then the condition of existence of the chaotic attractor changes to:
\[
\frac{\omega_2 - 1}{\tau_2 - 1 - \delta_1} < \frac{(\tau_1 - \delta_1 - \lambda_2)}{(\tau_1 - 1 - \delta_1)(\lambda_2 - \tau_2)},
\]

because \(\delta_1 = \delta_2\). Finally, the formulas (18), (20), and (22), and the inequalities (23), (24), and (25), or the inequalities (23), (25), and (26) if they are satisfied, determine rigorously the region for the parameter \(\alpha\) where the unified map (2) has robust homoclinic chaos.

## 4 Discussion

First, it is clear that the conditions of (23) are satisfied for all \(0 < \alpha < 1\). Second, it is difficult to solve rigorously the conditions for existence of the chaotic attractor (24) or (26) and its condition for stability (25) since these inequalities contain complicated square formulas. Hence, we use numerical estimates of the portion of the range \(0 \leq \alpha < 1\), for which robust homoclinic chaos occurs in the unified piecewise smooth map (2). Here we exclude the value \(\alpha = 0\), since there is no robust chaos in the Hénon map. We also exclude the value \(\alpha = 1\), since both fixed points given in (9) are not defined for this value.

Second, let us consider the critical curves corresponding to the conditions (24), (25), and (26) as follows:

\[
\begin{align*}
C_1 : & \quad \frac{\lambda_1 - 1}{\tau_1 - 1 - \delta_1} - \frac{\omega_2 - 1}{\tau_2 - 1 - \delta_2} = 0, \\
C_2 : & \quad (\tau_2 - \lambda_2) \lambda_1 + \tau_1 - \tau_2 - \delta_1 = 0, \\
C_3 : & \quad \frac{\omega_2 - 1}{\tau_2 - 1 - \delta_2} - \frac{\delta_1(\tau_2 - \delta_1 - \lambda_2)}{(\tau_1 - 1 - \delta_1)(\lambda_2 - \delta_1)} = 0,
\end{align*}
\]

From Fig.4 we remark that the curve \((C_2)\) has an intersection with the axis \(y = 0\), at \(\alpha = 0.0866592234\), then conditions (24) holds for \(\alpha \in [0, 0.0866592234]\), while the curve \((C_1)\) does not hits the axis \(y = 0\), then conditions (25) does not hold for all \(0 \leq \alpha < 1\), and the curve \((C_3)\) hits the axis \(y = 0\) also one time at \(\alpha = 0.493122734\), then condition (26) holds when \(\alpha \in [0.493122734, 1]\), where the Newton method for finding roots of an algebraic equation was used with an error of \(10^{-6}\). Thus, the homoclinic chaos presented by the unified chaotic map (2) is robust not stable when \(\alpha \in [0.493122734, 1]\), because the condition (25) is not hold in this interval. This chaotic attractor cannot be destroyed by small changes in the parameters. Since small changes in the parameters can only cause small
changes in the Lyapunov exponents. Hence, the percentage for the parameter $0 \leq \alpha < 1$, in which the map (2) converges to a robust chaotic attractor is approximately 50.688 percent, this result is also verified numerically by computing Lyapunov exponents and bifurcation diagram as shown in Fig. 2.

For $\alpha < 0.493122734$, the chaos is not robust in some ranges of the variable $\alpha$, because there are numerous small periodic windows as shown in Figs. 5 (a), 5 (b) for example the period-8 window at $\alpha = 0.025$. Also, for $\alpha = 0.114$, there is some periodic windows. We remark, also the existence of some regions in the $\alpha$–line where the largest Lyapunov exponent is positive, but this does not garantie the unicity of the attractor, contrary in the case where $\alpha \in [0.493122734, 1]$, where there is guarented that the attractor is unique, due to the analytical expressions (23), (24), (25), and (26).

When $\alpha$ approaches to 0, there is a break of smoothnessthe and the dynamics is too chaotic and presents some chaotic attractors very similar to the original Hénon attractor shown in Figs. 3 (a). Finally, it is interesting and surprising that the unified system (2) has such a property for an intermediate $\alpha$ while it was absent for $\alpha = 0$ or $\alpha = 1$. 

Figure 4: Critical curves corresponding to the conditions (24), (25) and (26).
Figure 5: (a) Variation of the Lyapunov exponents of the unified chaotic map (2) for $0.02 \leq \alpha \leq 0.03$. (b) The bifurcation diagram of the unified chaotic map (2) for $0.02 \leq \alpha \leq 0.03$, showing a period-8 attractor obtained for $\alpha = 0.025$. 
5 Conclusion

We have reported some results relevant to a new piecewise smooth 2-D discrete chaotic map as a unified chaotic system that contains the original Hénon and the Lozi systems as two extremes and other systems as a transition in between, and which has robust homoclinic chaos over a portion of its key system parameters, while this property is absent for the two systems at its extremes.

References


