About the boundedness of 3D continuous-time quadratic systems

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Abstract

In this paper, we generalize all the existing results in the current literature for the upper bound of a general 3-D quadratic continuous-time system. In particular, we find large regions in the bifurcation parameters space of this system where it is bounded.

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1 Introduction

Chaos in 3-D quadratic continuous-time autonomous systems was discovered in 1963 by E. Lorenz [1]. The boundedness of this system was the subject of many works. Bounded chaotic systems and the estimate of their bounds is important in chaos control, chaos synchronization, and their applications. The estimation of the upper bound of a chaotic system is quite difficult to achieve technically. In this work, we generalize all the relevant results of the literature and describe some of these bounds using multivariable function analysis.

The most general 3-D quadratic continuous-time autonomous system is given by

$$\begin{cases} x' = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz \\ y' = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5y^2 + b_6z^2 + b_7xy + b_8xz + b_9yz \\ z' = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 + c_6z^2 + c_7xy + c_8xz + c_9yz \end{cases} \quad (1)$$

where $(a_i, b_i, c_i)_{0\leq i\leq 9} \in \mathbb{R}^{30}$ are the bifurcation parameters. Several researchers have defined and studied quadratic 3-D chaotic systems as described in the references. The generalized Lorenz-like canonical form introduced in [6-11] gives a unique and unified classification for a large class of 3-D quadratic chaotic systems. This system contains all the well known quadratic systems given in [1-3-4-5-6]. In chaos control, chaos synchronization, and their applications, the estimation of an upper bound of the system under consideration is an important task. For example, in [23] the boundedness of the Lorenz system [1] was investigated and in [8] the boundedness of the Chen system [3] was investigated. Recently, a better upper bound for the Lorenz system for all positive values of its parameters was derived in [25], and it is the best result in the current literature because the estimation overcomes some problems related to the existence of singularities arising in the value of the upper bound given in [23].

In this paper, we generalize all these results concerning an upper bound for the general 3-D quadratic continuous-time autonomous system. In particular, we find large regions in the bifurcation parameter space of this system where it is bounded. The method is based on multivariable function analysis.

2 Estimate of the bound for the general system

To estimate the bound for the general system (1), we consider the function $V(x, y, z)$ defined by

$$V(x, y, z) = \frac{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}{2} \quad (2)$$

where $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ is any set of real constants for which the derivative of (2) along the solutions of (1) is given by

$$\frac{dV}{dt} = -\omega (x - \alpha_1)^2 - \varphi (y - \beta_1)^2 - \phi (z - \gamma_1)^2 + d \tag{3}$$

where

$$\begin{cases} d = \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 - \beta b_0 - \gamma c_0 - \alpha a_0 \\ \omega = \alpha a_4 - a_1 + \beta b_4 + \gamma c_4 \\ \varphi = \alpha a_5 - b_2 + \beta b_5 + \gamma c_5 \\ \phi = \alpha a_6 - c_3 + \beta b_6 + \gamma c_6 \\ \alpha_1 = \frac{a_0 - \alpha a_1 - \beta b_1 - \gamma c_1}{2\omega}, \text{ if } \omega \neq 0 \\ \beta_1 = \frac{b_0 - \alpha a_2 - \beta b_2 - \gamma c_2}{2\varphi}, \text{ if } \varphi \neq 0 \\ \gamma_1 = \frac{c_0 - \alpha a_3 - \beta b_3 - \gamma c_3}{2\phi}, \text{ if } \phi \neq 0. \end{cases} \tag{4}$$

Note that if $\omega = 0$ or $\varphi = 0$ or $\phi = 0$, then there is no need to calculate α_1, β_1 and γ_1, respectively, and a condition relating $(a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ to $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ is obtained. If not, we have the formulas given by the last three equalities of (4). The form of the function $\frac{dV}{dt}$ in (3) is possible if the following conditions on the coefficients $(a_i, b_i, c_i)_{0 \leq i \leq 9} \in \mathbb{R}^{30}$ hold:

$$\begin{cases} a_4 = 0, b_4 = -a_7, b_5 = 0, b_7 = -a_5, c_4 = -a_8, c_5 = -b_9 \\ c_6 = 0, c_7 = -a_9 - b_8, c_8 = -a_6, c_9 = -b_6 \\ b_1 = \alpha a_7 - \beta a_5 - a_2 + \gamma c_7 \\ c_1 = -a_3 - \gamma a_6 + \alpha a_8 + \beta b_8 \\ c_2 = \alpha a_9 - b_3 - \gamma b_6 + \beta b_9, \end{cases} \tag{5}$$

i.e., the system (1) becomes

$$\begin{cases} x' = a_0 + a_1 x + a_2 y + a_3 z + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 xz + a_9 yz \\ y' = b_0 + b_1 x + b_2 y + b_3 z - a_7 x^2 + b_6 z^2 - a_5 xy + b_8 xz + b_9 yz \\ z' = c_0 + c_1 x + c_2 y + c_3 z - a_8 x^2 - b_9 y^2 - (a_9 + b_8) xy - a_6 xz - b_6 yz \end{cases} \tag{6}$$

with the formulas for b_1, c_1, and c_2 given by the last three equations of (5).

To prove the boundedness of system (6), we assume that it is bounded and then we find its bound, i.e., assume that ω, φ, ϕ, and d are strictly positive, i.e.,

$$\begin{cases} \omega\alpha_1^2 + \varphi\beta_1^2 + \phi\gamma_1^2 - \beta b_0 - \gamma c_0 - \alpha a_0 > 0 \\ a_1 < \alpha a_4 + \beta b_4 + \gamma c_4 \\ b_2 < \alpha a_5 + \beta b_5 + \gamma c_5 \\ c_3 < \alpha a_6 + \beta b_6 + \gamma c_6. \end{cases} \tag{7}$$

Then if system (6) is bounded, the function (3) has a maximum value, and the maximum point (x_0, y_0, z_0) satisfies

$$\frac{(x_0-\alpha_1)^2}{\frac{d}{\omega}}+\frac{(y_0-\beta_1)^2}{\frac{d}{\varphi}}+\frac{(z_0-\gamma_1)^2}{\frac{d}{\phi}}=1. \tag{8}$$

Now consider the ellipsoid

$$\Gamma=\left\{(x,y,z)\in\mathbb{R}^3:\frac{(x-\alpha_1)^2}{\frac{d}{\omega}}+\frac{(y-\beta_1)^2}{\frac{d}{\varphi}}+\frac{(z-\gamma_1)^2}{\frac{d}{\phi}}=1,\omega,\varphi,\phi,d>0\right\}, \tag{9}$$

and define the function

$$\begin{cases} F(x,y,z)=G(x,y,z)+\lambda H(x,y,z) \\ G(x,y,z)=x^2+y^2+z^2 \\ H(x,y,z)=\frac{(x-\alpha_1)^2}{\frac{d}{\omega}}+\frac{(y-\beta_1)^2}{\frac{d}{\varphi}}+\frac{(z-\gamma_1)^2}{\frac{d}{\phi}}-1 \end{cases} \tag{10}$$

where $\lambda\in\mathbb{R}$ is a finite parameter. Then we have $\max_{(x,y,z)\in\Gamma} G=\max_{(x,y,z)\in\Gamma} F$ and

$$\begin{cases} \frac{\partial F(x,y,z)}{\partial x}=-2d^{-1}(\omega\lambda\alpha_1-(\omega\lambda+d)x) \\ \frac{\partial F(x,y,z)}{\partial y}=-2d^{-1}(\varphi\lambda\beta_1-(\varphi\lambda+d)y) \\ \frac{\partial F(x,y,z)}{\partial z}=-2d^{-1}(\phi\lambda\gamma_1-(\phi\lambda+d)z) \end{cases} \tag{11}$$

and the following cases according to the value of the parameter λ with respect to the values $-\frac{d}{\omega}, -\frac{d}{\varphi}$, and $\lambda\neq-\frac{d}{\phi}$ if $\omega,\varphi,\phi>0$. Otherwise, a similar study can be done.

(i) If $\lambda\neq-\frac{d}{\omega}, \lambda\neq-\frac{d}{\varphi}$, and $\lambda\neq-\frac{d}{\phi}$, then

$$(x_0,y_0,z_0)=\left(\frac{\omega\lambda\alpha_1}{d+\omega\lambda},\frac{\varphi\lambda\beta_1}{d+\varphi\lambda},\frac{\phi\lambda\gamma_1}{d+\phi\lambda}\right) \tag{12}$$

and

$$\max_{(x,y,z)\in\Gamma} G=\xi_1 \tag{13}$$

where

$$\xi_1=\frac{\omega^2\lambda^2\alpha_1^2}{(d+\omega\lambda)^2}+\frac{\varphi^2\lambda^2\beta_1^2}{(d+\varphi\lambda)^2}+\frac{\phi^2\lambda^2\gamma_1^2}{(d+\phi\lambda)^2}. \tag{14}$$

In this case, there exists a parameterized family (in λ) of bounds given by (14) of system (6).

(ii) If $\lambda = -\frac{d}{\omega}, (\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \varphi, \omega \neq \phi), \lambda \neq -\frac{d}{\varphi}, \lambda \neq -\frac{d}{\phi}$, then

$$(x_0, y_0, z_0) = \left(\pm\sqrt{\frac{d}{\omega}\left(1 - \frac{\xi_2}{\xi_3}\right)} + \alpha_1, \frac{-\beta_1\varphi}{\omega - \varphi}, \frac{-\gamma_1\phi}{\omega - \phi} \right) \tag{15}$$

where

$$\begin{cases} \xi_2 = \frac{\left(\varphi\beta_1^2 + \phi\gamma_1^2\right)(\omega-\phi)^2 d^3}{\omega^2} \\ \xi_3 = \frac{(\phi-\omega)^2(\omega-\varphi)^2 d^4}{\omega^4} \end{cases} \tag{16}$$

with the condition

$$\xi_3 \geq \xi_2. \tag{17}$$

This confirms that the value x_0 in (15) is well defined and the conditions $\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \varphi$, and $\omega \neq \phi$ are formulated as follows:

$$\begin{cases} a_1 \neq \alpha a_4 + \beta b_4 + \gamma c_4 \\ b_2 \neq \beta b_5 + \gamma c_5 + \alpha a_5 \\ c_3 \neq \beta b_6 + \gamma c_6 + \alpha a_6 \\ b_2 - a_1 \neq (a_5 - a_4)\,\alpha + (b_5 - b_4)\,\beta + (c_5 - c_4)\,\gamma \\ c_3 - a_1 \neq (a_6 - a_4)\,\alpha + (b_6 - b_4)\,\beta + (c_6 - c_4)\,\gamma. \end{cases} \tag{18}$$

In this case, we have

$$\max_{(x,y,z)\in\Gamma} G = \left(\sqrt{\frac{d}{\omega}\left(1 - \frac{\xi_2}{\xi_3}\right)} + \alpha_1 \right)^2 + \frac{\beta_1^2\varphi^2}{(\omega - \varphi)^2} + \frac{\gamma_1^2\phi^2}{(\omega - \phi)^2}. \tag{19}$$

(iii) If $\lambda \neq -\frac{d}{\omega}, \lambda = -\frac{d}{\varphi} (\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \varphi, \varphi \neq \phi), \lambda \neq -\frac{d}{\phi}$, then

$$(x_0, y_0, z_0) = \left(-\frac{\alpha_1\omega}{\varphi - \omega}, \pm\sqrt{\frac{d}{\varphi}\left(1 - \frac{\xi_4}{\xi_5}\right)} + \beta_1, \frac{\gamma_1\phi}{\phi - \varphi} \right) \tag{20}$$

where

$$\begin{cases} \xi_4 = \left(2\omega\varphi\phi\alpha_1^2 - 2\omega\varphi\phi\gamma_1^2 - \omega\varphi^2\alpha_1^2 - \omega\phi^2\alpha_1^2 + \omega^2\phi\gamma_1^2 + \varphi^2\phi\gamma_1^2\right)\varphi^2 \\ \xi_5 = (\phi - \varphi)^2 (\varphi - \omega)^2 d \end{cases} \tag{21}$$

with the condition

$$\xi_5 \geq \xi_4. \tag{22}$$

This confirms that the value y_0 in (20) is well defined and the conditions $\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \varphi$, and $\varphi \neq \phi$ are formulated by the first four equations of (18) and

$$c_3 - b_2 \neq (a_6 - a_5)\,\alpha + (b_6 - b_5)\,\beta + (c_6 - c_5)\,\gamma. \tag{23}$$

In this case, we have

$$\max_{(x,y,z)\in\Gamma} G = \left(\sqrt{\frac{d}{\varphi}\left(1 - \frac{\xi_4}{\xi_5}\right)} + \beta_1\right)^2 + \frac{\alpha_1^2\omega^2}{(\varphi-\omega)^2} + \frac{\gamma_1^2\phi^2}{(\varphi-\phi)^2}. \tag{24}$$

(iv) If $\lambda \neq -\frac{d}{\omega}, \lambda \neq -\frac{d}{\varphi}, \lambda = -\frac{d}{\phi}\left(\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \phi, \phi \neq \varphi\right),$ then

$$(x_0, y_0, z_0) = \left(\frac{-\alpha_1\omega}{\phi-\omega}, \frac{-\beta_1\varphi}{\phi-\varphi}, \pm\sqrt{\frac{d}{\phi}\left(1 - \frac{\xi_6}{\xi_7}\right)} + \gamma_1\right) \tag{25}$$

where

$$\begin{cases} \xi_6 = \left(\omega\varphi^2\alpha_1^2 - 2\omega\varphi\phi\beta_1^2 - 2\omega\varphi\phi\alpha_1^2 + \omega\phi^2\alpha_1^2 + \omega^2\varphi\beta_1^2 + \varphi\phi^2\beta_1^2\right)\phi^2 \\ \xi_7 = (\phi-\varphi)^2(\phi-\omega)^2 d \end{cases} \tag{26}$$

with the condition

$$\xi_7 \geq \xi_6. \tag{27}$$

This confirms that the value z_0 in (25) is well defined and the conditions $\omega \neq 0, \varphi \neq 0, \phi \neq 0, \omega \neq \phi$, and $\phi \neq \varphi$ are formulated by the first four equations of (18) and (23), respectively.

The other possible cases are treated using the same logic.

Theorem 1 *Assume that conditions (4), (5), and (7) hold. Then the general 3-D quadratic continuous-time system (1) is bounded, i.e., it is contained in the ellipsoid (9).*

Similar results can be found using the cases discussed above.

3 Example

Consider the Lorenz system given by

$$\begin{cases} x' = a(y-x) \\ y' = cx - y - xz \\ z' = xy - bz, \end{cases} \tag{28}$$

i.e., $a_i = 0, i = 0,3,4,5,6,7,8,9, a_1 = -a, a_2 = a, b_i = 0, i = 0,3,4,5,6,7,9, b_1 = c, b_2 = -1, b_8 = -1$, $c_i = 0, i = 0,1,2,4,5,6,8,9, c_3 = -b$, and $c_7 = 1$. We choose $\alpha = \beta = 0$ and $\gamma = a+c$ as in [25]. Thus $V(x,y,z) = \frac{x^2+y^2+(z-(a+c))^2}{2}$ and $d = b\left(\frac{a+c}{2}\right)^2, \omega = a, \varphi = 1, \phi = b, \alpha_1 = 0, \beta_1 = 0$, and $\gamma_1 = \frac{a+c}{2}$. Then we have $\frac{dV}{dt} = -ax^2 - y^2 - b\left(z - \frac{a+c}{2}\right)^2 + b\left(\frac{a+c}{2}\right)^2$, which is the same as in [25]. Also, it is easy to verify that conditions (5) and (7) hold for this case. The ellipsoid Γ is given by $\Gamma = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{\frac{b}{a}\left(\frac{a+c}{2}\right)^2} + \frac{y^2}{b\left(\frac{a+c}{2}\right)^2} + \frac{\left(z-\frac{a+c}{2}\right)^2}{\left(\frac{a+c}{2}\right)^2} = 1, a,b,c > 0\right\}$, which is also the same as in [25]. Finally, we have the result shown in [25] that confirms that if $a > 0$, $b > 0$, and $c > 0$, then the Lorenz system [1] is contained in the sphere $\Omega = \left\{(x,y,z) \in \mathbb{R}^3 \;/\; x^2 + y^2 + (z-a-c)^2 = R^2\right\}$, where

$$R^2 = \begin{cases} \frac{(a+c)^2 b^2}{4(b-1)}, & \text{if } a \geq 1, b \geq 2 \\ (a+c)^2, & \text{if } a > \frac{b}{2}, b < 2 \\ \frac{(a+c)^2 b^2}{4a(b-a)}, & \text{if } a < 1, b \geq 2a. \end{cases} \tag{29}$$

4 Conclusion

Using multivariable function analysis, we generalize all the results about finding an upper bound for the general 3-D quadratic continuous-time autonomous system. In particular, we find large regions in the bifurcation parameters space of this system where it is bounded.

References

[1] E. N. Lorenz, Deterministic nonperiodic flow, J. Atoms. Sci. 20 (1963) 130–141.

[2] O. E. Rössler, An equation for continuous chaos, Phys. Lett. A 57 (1976) 397–398.

[3] G. Chen, T. Ueta, Yet another chaotic attractor, Int. J. Bifur. Chaos 9 (1999) 1465–1466.

[4] J. H. Lu, G. Chen, A new chaotic attractor coined, Int. J. Bifur. Chaos 12 (2002) 659–661.

[5] J. H. Lu, G. Chen, D. Cheng, S. Celikovsky, Bridge the gap between the Lorenz system and the Chen system, Int. J. Bifur. Chaos 12 (2002) 2917–2926.

[6] Q. Yang, G. Chen, T. Zhou, A unified Lorenz-type system and its canonical form, Int. J. Bifur. Chaos 16 (2006) 2855–2871.

[7] G. Qi, G. Chen, S. Du, Analysis of a new chaotic system, Physica A 352 (2005) 295–308.

[8] T. Zhou, Y. Tang, G. Chen, Complex dynamical behaviors of the chaotic Chen's system, Int. J. Bifur. Chaos 9 (2003) 2561–2574.

[9] T. Zhou, G. Chen, Q. Yang, Constructing a new chaotic system based on the Sh'ilnikov criterion, Chaos, Solitons and Fractals 19 (2004) 985–993.

[10] T. Zhou, G. Chen, Classification of chaos in 3-D autonomous quadratic systems-I. Basic framework and methods, Int. J. Bifur. Chaos 16 (2006) 2459–2479.

[11] S. Celikovsky, G. Chen, On a generalized Lorenz canonical form of chaotic systems, Int. J. Bifur. Chaos 12 (2002), 1789–1812.

[12] G. Chen, Controlling Chaos and Bifurcations in Engineering Systems, CRC Press, Boca Raton, FL (1999).

[13] J. C. Sprott, Some simple chaotic flows, Phys. Rev. E 50 (1994) 647–650.

[14] J. C. Sprott, Automatic generation of strange attractors, Comput. & Graphics 17 (1993) 325–332.

[15] J. C. Sprott, S. J. Linz, Algebraically simple chaotic flows, J. Chaos Theory App. 5 (2000) 3–22.

[16] W. G. Hoover, Remark on Some simple chaotic flows, Phys. Rev. E 51 (1995) 759–760.

[17] H. A. Posch, W. G. Hoover, F. J. Vesely, Canonical dynamics of the Nosé oscillator: Stability, order, and chaos, Phys. Rev. A 33 (6) (1986) 4253–4265.

[18] J. C. Sprott, Simplest dissipative chaotic flow. Phys. Lett. A 228 (1997) 271–274.

[19] S. J. Linz, J. C. Sprott, Elementary chaotic flow. Phys. Lett. A 259 (1999) 240–245.

[20] Z. Elhadj, Analysis of a new chaotic system with three quadratic nonlinearities, Dynamics of Continuous, Discrete, and Impulsive Systems B 14 (2007) 603–613.

[21] Z. Elhadj, Dynamical analysis of a 3-D chaotic system with only two quadratic nonlinearities, J. Systems Science and Complexity 21 (2008) 67–75.

[22] W. B. Liu, G. Chen, A new chaotic system and its generation, Int. J. Bifur. Chaos 13 (2003) 261–267.

[23] G. Leonov, A. Bunin, N. Koksch, Attractor localization of the Lorenz system, Zeitschrift fur Angewandte Mathematik und Mechanik 67 (1987) 649–656.

[24] A. Y. Pogromsky, G. Santoboni, H. Nijmeijer, An ultimate bound on the trajectories of the Lorenz systems and its applications, Nonlinearity 16 (2003) 1597–1605.

[25] D. Li, J. A. Lu, X. Wu, G. Chen, Estimating the bounds for the Lorenz family of chaotic systems, Chaos, Solitons and Fractals 23 (2005) 529–534.