Quadratic maps of the plane: Tutorial and review

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Abstract

In addition to some new fundamental results about the dynamics of general 2-D quadratic maps, this paper offers an overview of some important issues related to the general case of these mappings, especially predicting the strange attractors and their properties.

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1 Introduction

The most general 2-D quadratic map is given by

$$\begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 + a_5 x_k y_k = f(x_k, y_k) \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 + b_4 y_k^2 + b_5 x_k y_k = g(x_k, y_k), \end{cases} \quad (1)$$

where $(a_i, b_i)_{1 \leq i \leq 6} \in \mathbb{R}^{12}$ are the bifurcations parameters. Some special cases of the map (1) can be used in potential applications in several different ways and types of studies [20-21-22]. Some important results about the dynamical properties, bifurcations, and stability of some special cases of the 2-D map (1) are given in [1-4-5-6-7-8]. However, there are a few papers that focus on the general case of this map. For example, in [12] some solutions of low-dimensional, low-order polynomial maps were classified numerically as either fixed point, limit cycle, chaotic, or unstable using Lyapunov exponent calculations, with the result that a few percent are chaotic. For the 2-D quadratic maps, this percentage is about $11.10 \pm 0.36\%$. Furthermore, in [13] the correlation dimension was calculated for the strange attractors obtained numerically for some cases of the map (1), and it was found that the average correlation dimension scales approximately as the square root of the dimension of the system with a small variation. In [10-14] a systematic search for chaotic orbits of the general 2-D quadratic map (1) with randomly chosen coefficients was given using a simple computer program that gives different attractors. Some simple special cases of the general 2-D quadratic map (1) were studied in detail in [1-2-3-9-17-18-19], with analytical results in [1-2-3].

This paper is organized as follows. In the following section, we discuss the equivalence between the elements of the general map (1) in order to reduce the number of possible chaotic attractors. The invertibility of the map (1) is discussed in Section 2. The existence of unbounded and bounded orbits is investigated analytically in Section 3, and some criteria for seeking chaotic attractors are listed in Section 4. Analytical predictions of some system orbits of the general map (1) are given in Section 5. In Section 6 a classification of the possible chaotic orbits is given according to the number of nonlinearities, showing how to reduce all the dynamics of the general case (1) to a finite number of maps with well known formulas. Several numerical results given in the Section 7 confirm the theory and show new features in the dynamics of the map (1). The final section concludes the paper.

2 Equivalences in the general 2-D quadratic maps

In this section, we discuss in some detail, with examples, the equivalence relations between the elements of the map (1). The analysis shows that it is possible for a map with $1 \leq m \leq 6$ quadratic nonlinearities to be equivalent to a map with all $1 \leq m \leq 6$ quadratic nonlinearities. First, interchanging

x and y produces two kinds of equivalent maps. Thus, one has the following theorem:

Theorem 1 *The two maps*

$$\left(\begin{array}{c} x_{k+1}=f\left(x_k,y_k\right) \\ \\ y_{k+1}=g\left(x_k,y_k\right) \end{array}\right),\left(\begin{array}{c} x_{k+1}=g\left(y_k,x_k\right) \\ \\ y_{k+1}=f\left(y_k,x_k\right) \end{array}\right) \tag{2}$$

are topologically equivalent.

This theorem facilitates study of the map (1) since we want to classify the attractors of the map (1) according to the number of nonlinearities. Since there is 6 nonlinearities in the map (1), the number of maps with $1 \leq m \leq 6$ nonlinearities is $2N_m$, but due to this equivalence criterion, this number will be reduced to N_m. On the other hand, if we consider two different quadratic maps given by

$$f_1\left(x,y\right)=\left(\begin{array}{c} a_0+a_1x+a_2y+a_3x^2+a_4y^2+a_5xy \\ \\ b_0+b_1x+b_2y+b_3x^2+b_4y^2+b_5xy \end{array}\right) \tag{3}$$

$$g_1\left(x,y\right)=\left(\begin{array}{c} c_0+c_1x+c_2y+c_3x^2+c_4y^2+c_5xy \\ \\ d_0+d_1x+d_2y+d_3x^2+d_4y^2+d_5xy \end{array}\right), \tag{4}$$

then we say that f_1 and g_1 are topologically equivalent if there exists a homeomorphism h such that

$$g_1\circ h\left(x,y\right)=h\circ f_1\left(x,y\right), \text{for all } \left(x,y\right)\in\mathbb{R}^2. \tag{5}$$

For simplicity we choose h as the linear homeomorphism defined by

$$h\left(x,y\right)=\left(\begin{array}{cc} e_1 & e_2 \\ l_1 & l_2 \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right) \tag{6}$$

with the condition

$$d=e_1l_2-e_2l_1\neq 0. \tag{7}$$

After some tedious calculations, the expressions for the coefficients $(c_i,d_i)_{0\leq i\leq 5}$ of the map (4) in terms of the coefficients $(a_i,b_i)_{0\leq i\leq 5}$ of the map (3) are given by:

$$\begin{cases} c_0 = e_1 a_0 + e_2 b_0 \\ c_1 = \frac{e_1 a_1 l_2 - e_1 a_2 l_1 + e_2 b_1 l_2 - e_2 b_2 l_1}{e_1 l_2 - e_2 l_1} \\ c_2 = \frac{e_1 e_2 b_2 - e_1 e_2 a_1 + e_1^2 a_2 - e_2^2 b_1}{e_1 l_2 - e_2 l_1} \\ c_3 = \frac{e_1 a_3 l_2^2 - e_2 b_5 l_1 l_2 - e_1 a_5 l_1 l_2 + e_1 a_4 l_1^2 + e_2 b_3 l_2^2 + e_2 b_4 l_1^2}{(e_1 l_2 - e_2 l_1)^2} \\ c_4 = \frac{-\left(e_1^2 e_2 a_5 - e_2^3 b_3 - e_1 e_2^2 a_3 - e_1^3 a_4 - e_1^2 e_2 b_4 + e_1 e_2^2 b_5\right)}{(e_1 l_2 - e_2 l_1)^2} \\ c_5 = \frac{-\left(2 e_1 e_2 a_3 l_2 - e_1 e_2 a_5 l_1 + 2 e_1 e_2 b_4 l_1 - e_1 e_2 b_5 l_2 + 2 e_1^2 a_4 l_1 - e_1^2 a_5 l_2 + 2 e_2^2 b_3 l_2 - e_2^2 b_5 l_1\right)}{(e_1 l_2 - e_2 l_1)^2} \end{cases} \tag{8}$$

and

$$\begin{cases} d_0 = l_1 a_0 + l_2 b_0 \\ d_1 = \frac{a_1 l_1 l_2 - b_2 l_1 l_2 - a_2 l_1^2 + b_1 l_2^2}{e_1 l_2 - e_2 l_1} \\ d_2 = \frac{e_1 a_2 l_1 - e_2 a_1 l_1 + e_1 b_2 l_2 - e_2 b_1 l_2}{e_1 l_2 - e_2 l_1} \\ d_3 = \frac{-\left(a_5 l_1^2 l_2 - b_3 l_2^3 - a_3 l_1 l_2^2 - a_4 l_1^3 - b_4 l_1^2 l_2 + b_5 l_1 l_2^2\right)}{(e_1 l_2 - e_2 l_1)^2} \\ d_4 = \frac{e_1^2 a_4 l_1 - e_1 e_2 b_5 l_2 - e_1 e_2 a_5 l_1 + e_2^2 a_3 l_1 + e_1^2 b_4 l_2 + e_2^2 b_3 l_2}{(e_1 l_2 - e_2 l_1)^2} \\ d_5 = \frac{-\left(2 e_2 a_3 l_1 l_2 - e_1 a_5 l_1 l_2 + 2 e_1 b_4 l_1 l_2 - e_2 b_5 l_1 l_2 + 2 e_1 a_4 l_1^2 + 2 e_2 b_3 l_2^2 - e_2 a_5 l_1^2 - e_1 b_5 l_2^2\right)}{(e_1 l_2 - e_2 l_1)^2}. \end{cases} \tag{9}$$

According to the values of $e_1, e_2, l_1,$ and l_2 and the condition (7), there exists an infinity of linear transformations that convert the general quadratic map of the plane (1) to another one of the form (4). Hence, we have proved the following theorem:

Theorem 2 *If (7) holds and the vector* $(c_i, d_i)_{1 \leq i \leq 6} \in \mathbb{R}^{12}$ *has the values given in (8) and (9), then the map (1) with coefficients* $(a_i, b_i)_{1 \leq i \leq 6} \in \mathbb{R}^{12}$ *is topologically equivalent to itself with the coefficients* $(c_i, d_i)_{1 \leq i \leq 6} \in \mathbb{R}^{12}$.

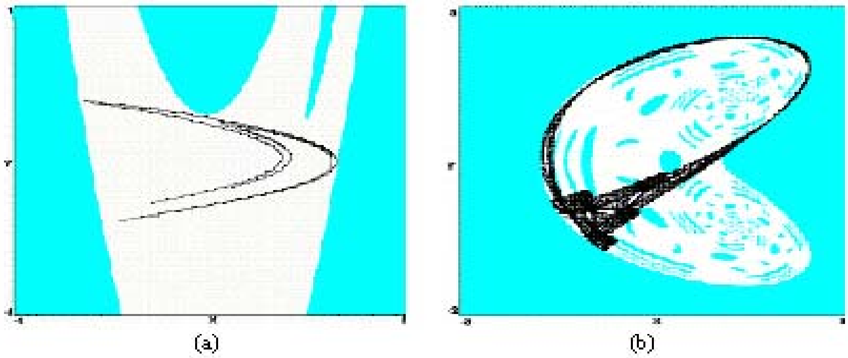

Figure 1: (a) Chaotic attractor of the Hénon map with $a_3 = -1.4$ and $a_2 = 0.3$. (b) Chaotic attractor of the map (14) with $a_4 = 0.59948$ and $a_1 = 1$.

Note that if $e_1 = 0, e_2 = 1, l_1 = 1$, and $l_2 = 0$, we get Theorem 1. If $e_1 = 0, e_2 = 1, l_1 = -1$, and $l_2 = 0$, then the resulting attractors are rotated 90 degrees counterclockwise from the original given by the map (1). If $e_1 = -1, e_2 = 0, l_1 = 0$, and $l_2 = -1$, then the resulting attractors are rotated through 180 degrees, and through 270 degrees if $e_1 = 0, e_2 = -1, l_1 = -1$, and $l_2 = 0$.

As a test for Theorem 2, consider the Hénon map [2] given by

$$\begin{pmatrix} x_{k+1} = 1 + a_2 y_k + a_3 x_k^2 \\ \\ y_{k+1} = x_k \end{pmatrix} \tag{10}$$

as shown in Fig. 1(a).

Then all maps of the form

$$\begin{pmatrix} x_{k+1} = e_1 + \frac{(-e_1a_2l_1+e_2l_2)x_k}{e_1l_2-e_2l_1} + \frac{\left(e_1^2a_2-e_2^2\right)y_k}{e_1l_2-e_2l_1} + \frac{e_1l_2^2a_3x_k^2}{(e_1l_2-e_2l_1)^2} + \frac{e_1e_2^2a_3y_k^2}{(e_1l_2-e_2l_1)^2} - \frac{2e_1e_2l_2a_3x_ky_k}{(e_1l_2-e_2l_1)^2} \\ \\ y_{k+1} = l_1 + \frac{-a_2l_1^2+l_2^2}{e_1l_2-e_2l_1}x_k + \frac{e_1a_2l_1-e_2l_2}{e_1l_2-e_2l_1}y_k + \frac{l_1l_2^2a_3x_k^2}{(e_1l_2-e_2l_1)^2} + \frac{e_2^2l_1a_3y_k^2}{(e_1l_2-e_2l_1)^2} - \frac{2e_2a_3l_1l_2x_ky_k}{(e_1l_2-e_2l_1)^2} \end{pmatrix} \tag{11}$$

are topologically equivalent to the Hénon map (10). As a verification of the formulas (11) we choose $e_1 = 1, e_2 = 0, l_1 = 0$, and $l_2 = 1$, i.e., the identity transformation, for which rigorous substitutions in the map (11) gives the Hénon map (10). Note that any quadratic planar map with a constant determinant of the Jacobian matrix can be put into the form of the Hénon map [1-2] by a linear coordinate transformation. For example, the parameters $(a_i, b_i)_{1\leq i\leq 6} \in \mathbb{R}^{12}$ lead to the following conditions:

$$\begin{array}{c} a_3b_5 = b_3a_5 \\ a_5b_4 = a_4b_5 \\ a_3b_4 = a_4b_3 \\ a_1b_2 - a_2b_1 \neq 0 \\ 2a_3b_2 - 2a_2b_3 + a_1b_5 - b_1a_5 = 0 \\ 2a_1b_4 - 2b_1a_4 - a_2b_5 + b_2a_5 = 0. \end{array} \tag{12}$$

Indeed, the determinant of the Jacobian matrix of the map (11) has the constant value $(-a_2)$. On the one hand, Theorem 2 indicates that it is possible that a quadratic map with m nonlinearities is topologically equivalent to another map with j nonlinearities, where $1 \leq m, j \leq 6$. For example, the Hénon map (10) is topologically equivalent to a map with one nonlinearity x^2 if $e_1l_2 \neq 0$, $l_1 = 0$, and $e_2 = 0$, and to a map with six nonlinearities if $e_1 \neq 0, e_2 \neq 0, l_1 \neq 0, l_2 \neq 0$, and $e_1l_2 - e_2l_1 \neq 0$, and so on. On the other hand, all maps of the form

$$\left(\begin{array}{c} x_{k+1} = e_1 + \frac{(e_1a_1l_2+e_2l_2)x_k}{e_1l_2-e_2l_1} - \frac{\left(e_1e_2a_1+e_2^2\right)y_k}{e_1l_2-e_2l_1} + \frac{e_1a_4l_1^2x_k^2}{(e_1l_2-e_2l_1)^2} + \frac{e_1^3a_4y_k^2}{(e_1l_2-e_2l_1)^2} - \frac{2e_1^2a_4l_1x_ky_k}{(e_1l_2-e_2l_1)^2} \\ \\ y_{k+1} = l_1 + \frac{\left(a_1l_1l_2+l_2^2\right)x_k}{e_1l_2-e_2l_1} - \frac{(e_2a_1l_1+e_2l_2)y_k}{e_1l_2-e_2l_1} + \frac{a_4l_1^3x_k^2}{(e_1l_2-e_2l_1)^2} + \frac{e_1^2a_4l_1y_k^2}{(e_1l_2-e_2l_1)^2} - \frac{2e_1a_4l_1^2x_ky_k}{(e_1l_2-e_2l_1)^2} \end{array} \right) \tag{13}$$

are topologically equivalent to the following map given in [17]

$$\left(\begin{array}{c} x_{k+1} = 1 + a_1x_k + a_4y_k^2 \\ \\ y_{k+1} = x_k \end{array} \right). \tag{14}$$

As a verification of the formulas (13) we choose $e_1 = 1, e_2 = 0, l_1 = 0$, and $l_2 = 1$, i.e., the identity transformation, which gives by rigorous substitutions in the map (13) the formula for the map (14). A chaotic attractor of the map (14) is shown in Fig. 1(b).

3 The invertibility of the map

A map is invertible if the determinant δ of its Jacobian matrix is non-zero for all state variables of the system. For the map (1), the Jacobian matrix is given by:

$$J(x,y) = \begin{pmatrix} a_1 + 2a_3x + a_5y & a_2 + 2a_4y + a_5x \\ b_1 + 2b_3x + b_5y & b_2 + 2b_4y + b_5x \end{pmatrix}. \tag{15}$$

For the map (1) one has

$$\delta = \xi_1 x^2 + \xi_2 y^2 + \xi_3 xy + \xi_4 x + \xi_5 y + \xi_6, \tag{16}$$

where

$$\begin{cases} \xi_1 = 2a_3b_5 - 2b_3a_5 \\ \xi_2 = 2a_5b_4 - 2a_4b_5 \\ \xi_3 = 4a_3b_4 - 4a_4b_3 \\ \xi_4 = 2a_3b_2 - 2a_2b_3 + a_1b_5 - b_1a_5 \\ \xi_5 = 2a_1b_4 - 2b_1a_4 - a_2b_5 + b_2a_5 \\ \xi_6 = a_1b_2 - a_2b_1. \end{cases} \tag{17}$$

Thus, the map (1) is invertible if and only if $\xi_1 x^2 + \xi_2 y^2 + \xi_3 xy + \xi_4 x + \xi_5 y + \xi_6 \neq 0$, for all $(x,y) \in \mathbb{R}^2$, and this possible if and only if $\xi_1 x^2 + \xi_2 y^2 + \xi_3 xy + \xi_4 x + \xi_5 y + \xi_6 > 0$ or $\xi_1 x^2 + \xi_2 y^2 + \xi_3 xy + \xi_4 x + \xi_5 y + \xi_6 < 0$ for all $(x,y) \in \mathbb{R}^2$. Thus one has the following theorem:

Theorem 3 *The map (1) is invertible if and only if one of the following conditions holds:*

$$\begin{cases} \xi_1 > 0, \xi_3^2 - 4\xi_1\xi_2 < 0 \\ 4\xi_1\xi_2\xi_6 + \xi_3\xi_4\xi_5 - \xi_1\xi_5^2 - \xi_2\xi_4^2 - \xi_3^2\xi_6 > 0 \end{cases} \tag{18}$$

or

$$\begin{cases} \xi_1 < 0, \xi_3^2 - 4\xi_1\xi_2 < 0 \\ 4\xi_1\xi_2\xi_6 + \xi_3\xi_4\xi_5 - \xi_1\xi_5^2 - \xi_2\xi_4^2 - \xi_3^2\xi_6 > 0. \end{cases} \tag{19}$$

An intermediate result of this theorem is that if conditions (12) hold, then the Hénon family [1-2] is invertible. Two important questions arise in this situation. The first concerns the cases where the inverse map is quadratic, and the second is where the inverse map is conjugate to the original map. It is well known that the Hénon map satisfies these criteria [1-2]. The investigation of these two questions is not an easy problem because it depends on the solvability of the two equations $f(x, y) = z$ and $g(x, y) = u$, where (x, y) are the coordinates of the map (1) and (z, u) are the coordinates of its inverse map in cases where it exists. For the investigation of some special cases see [4-5-7-8].

4 Existence of unbounded and bounded orbits

In this section, we investigate domains for the parameters $(a_i, b_i)_{1\leq i\leq 6} \in \mathbb{R}^{12}$ in which the map (1) has unbounded or bounded orbits. We use the idea of the non-existence of fixed points, because if there is no fixed point, then there is no chaos in the map (1) as shown in the next subsection. Note that in this paper we use the following simple result available in most textbooks on linear algebra:

Theorem 4 *The polynomial $Ax^2 + Bx + C$ has no real zeros if and only if $A > 0$ and $B^2 - 4AC < 0$, or $A < 0$ and $B^2 - 4AC < 0$.*

4.1 Existence of unbounded orbits

First, a fixed point (x, y) of the map (1) must simultaneously satisfy the following two equalities:

$$C_0 : \begin{cases} a_0 + a_1x + a_2y + a_3x^2 + a_4y^2 + a_5xy = x \\ b_0 + b_1x + b_2y + b_3x^2 + b_4y^2 + b_5xy = y. \end{cases} \tag{20}$$

Second, we note that most special cases of the map (1) are unbounded. In this subsection, we give sufficient conditions for the existence of these unbounded orbits. Indeed, if there are no fixed points, then one has from (20) that the polynomials $f(x,y)-x$ or $g(x,y)-y$ are either positive or negative for all $(x,y)\in\mathbb{R}^2$. Assume for example that $f(x,y)-x$ is positive, and let $x_0\geq 0$. Then one has for all integer k that $f(x_k,y_k)>x_k$, i.e., $x_{k+1}>x_k>x_{k-1}>....>x_0\geq 0$. Let us consider the Euclidean distance $d(x_k,0)=x_k$ that measures the distance between the first component x_k of the map (1) and the origin 0 on the real line. Then we have $d(x_{k+1},0)>d(x_k,0)>d(x_{k-1},0)>....>d(x_0,0)\geq 0$. Hence there a real number $\triangle>0$ such that $d(x_k,0)=d(x_{k-1},0)+\triangle$, which implies that $d(x_k,0)=d(x_0,0)+(k+1)\triangle$. Finally, one has $\lim_{k\to+\infty}d(x_k,0)=+\infty$. If $x_0<0$, the same logic applies. When there are fixed points, there are domains that contain all bounded orbits, i.e., possibly chaotic attractors. On the other hand, chaotic attractors in a system without fixed points are probably rare, but it is possible for a system to have a chaotic attractor without having fixed points. There is one case of a conservative system that is chaotic without any fixed points [11].

The idea of the non-existence of fixed points for the map (1) is used to prove the following theorem:

Theorem 5 *If $(a_i,b_i)_{1\leq i\leq 6}\in\Omega=\cup_{i=1}^{i=4}C_i\subset\mathbb{R}^{12}$, then all orbits of the map (1) are unbounded,*

where

$$C_1:\begin{cases} a_3>\frac{a_5^2}{4a_4},\ a_4>0 \\ a_0>\frac{\left(2a_1-1-a_1^2\right)a_4+(a_1-1)a_2a_5-a_2^2a_3}{a_5^2-4a_3a_4}\end{cases}\tag{21}$$

$$C_2:\begin{cases} a_3<\frac{a_5^2}{4a_4},\ a_4<0 \\ a_0>\frac{\left(2a_1-1-a_1^2\right)a_4+(a_1-1)a_2a_5-a_2^2a_3}{a_5^2-4a_3a_4}\end{cases}\tag{22}$$

$$C_3:\begin{cases} b_3>\frac{b_5^2}{4b_4},\ b_4>0 \\ b_0>\frac{b_1^2b_4-\left(2b_2-b_2^2-1\right)b_3-(b_2-1)b_1b_5}{4b_3b_4-b_5^2}\end{cases}\tag{23}$$

$$C_4 : \begin{cases} b_3 < \frac{b_5^2}{4b_4}, b_4 < 0 \\ b_0 > \frac{b_1^2 b_4 - \left(2b_2 - b_2^2 - 1\right) b_3 - (b_2 - 1) b_1 b_5}{4b_3 b_4 - b_5^2}. \end{cases} \quad (24)$$

Proof. The map (1) has no fixed points if one of the following inequalities holds for all $(x, y) \in \mathbb{R}^2$:

(1) $a_3 x^2 + (a_1 - 1 + a_5 y) x + a_4 y^2 + a_2 y + a_0 > 0$
(2) $a_3 x^2 + (a_1 - 1 + a_5 y) x + a_4 y^2 + a_2 y + a_0 < 0$
(3) $b_3 x^2 + (b_1 + b_5 y) x + b_4 y^2 + (b_2 - 1) y + b_0 > 0$
(4) $b_3 x^2 + (b_1 + b_5 y) x + b_4 y^2 + (b_2 - 1) y + b_0 < 0$.

The discriminant of the first case is given by $d = (a_5^2 - 4a_3 a_4) y^2 + (2a_1 a_5 - 4a_2 a_3 - 2a_5) y - 4a_0 a_3 + a_1^2 - 2a_1 + 1$, and so the inequality $a_3 x^2 + (a_1 - 1 + a_5 y) x + a_4 y^2 + a_2 y + a_0 > 0$ holds for all $(x, y) \in \mathbb{R}^2$ if and only if $d < 0$ for all $y \in \mathbb{R}$, and $a_3 > 0$, i.e., $(1 + a_1^2 - 2a_1) a_4 + (1 - a_1) a_2 a_5 + (a_5^2 - 4a_3 a_4) a_0 + a_2^2 a_3 < 0$ and $a_5^2 - 4a_3 a_4 < 0$, and this is possible if (21) holds. The other cases are obtained using the same logic. ■

4.2 Existence of bounded orbits

Let us define the following subsets of $\mathbb{R}^{12}$:

$$\begin{cases} C_{11} : a_3 < \frac{a_5^2}{4a_4} \\ C_{12} : a_4 < 0 \\ C_{13} : a_0 < \frac{\left(2a_1 - 1 - a_1^2\right) a_4 + (a_1 - 1) a_2 a_5 - a_2^2 a_3}{a_5^2 - 4a_3 a_4} \end{cases} \quad (25)$$

$$\begin{cases} C_{21} : a_3 > \frac{a_5^2}{4a_4} \\ C_{22} : a_4 > 0 \\ C_{23} : a_0 < \frac{\left(2a_1 - 1 - a_1^2\right) a_4 + (a_1 - 1) a_2 a_5 - a_2^2 a_3}{a_5^2 - 4a_3 a_4} \end{cases} \quad (26)$$

$$\begin{cases} C_{31} : b_3 < \frac{b_5^2}{4b_4} \\ C_{32} : b_4 < 0 \\ C_{33} : b_0 < \frac{b_1^2 b_4 - \left(2b_2 - b_2^2 - 1\right) b_3 - (b_2 - 1) b_1 b_5}{4b_3 b_4 - b_5^2} \end{cases} \quad (27)$$

$$\begin{cases} C_{41} : b_3 > \frac{b_5^2}{4b_4} \\ C_{42} : b_4 > 0 \\ C_{43} : b_0 < \frac{b_1^2 b_4 - \left(2b_2 - b_2^2 - 1\right) b_3 - (b_2 - 1) b_1 b_5}{4b_3 b_4 - b_5^2}. \end{cases} \quad (28)$$

Thus one has:

$$\bar{C}_i = \cup_{j=1}^{j=3} C_{ij}, i = 1, 2, 3, 4. \quad (29)$$

Here, the subsets $\left(\bar{C}_i\right)_{1 \leq i \leq 4}$ are the complements in $\mathbb{R}^{12}$ of the subsets $\left(C_i\right)_{1 \leq i \leq 4}$ given in (21), (22), (23), and (24). As a result, we have the following theorem:

Theorem 6 *If* $(a_i, b_i)_{1 \leq i \leq 6} \in \bar{\Omega} = \cap_{i=1}^{i=4} \cup_{j=1}^{j=3} C_{ij} \subset \mathbb{R}^{12}$, *then the map (1) has possible bounded orbits,*

where $\bar{\Omega}$ is the complement in $\mathbb{R}^{12}$ of the set Ω.

Because the map (1) is quadratic, the possible number of fixed points is from 1 to 4, and thus one has the following theorem:

Theorem 7 *For the map (1) there at least four possible topologically different bounded attractors.*

5 Some criteria for finding chaotic orbits

Roughly speaking, chaotic behavior implies sensitive dependence on initial conditions, with at least one positive Lyapunov exponent. From some well known special cases of the 2-D quadratic map (1), several remarks are in order:

1. Two of the 12 coefficients can be set to 1 by renormalizing x and y. We use this criterion for each case separately.

2. Because there must be at least one nonlinearity for chaos, we classify the special maps of (1) according to their number of nonlinearities. We found 30 cases, some of which are topologically equivalent by the above analysis.

3. The function f must depend on y, and g must depend on x if the map (1) is a truly 2-D map.

4. For bounded orbits, the absolute value of the determinant (averaged along the orbit) of the Jacobian matrix is not greater than unity.

5. There must be an expanding direction for chaos.

6. We choose a classification according to the number of nonlinearities, with the simplest examples having the maximum number of vanishing coefficients.

6 Analytical prediction of system orbits

This section includes some theorems that determine rigorously the domains for the parameters $(a_i, b_i)_{1\le i\le 6} \in \mathbb{R}^{12}$ in which the orbits of the map (1) are asymptotically stable and possibly chaotic. Generally, chaos can occur in the map (1) if it has at least one fixed point that is not asymptotically stable, i.e., it must be a saddle or unstable fixed point.

First, consider a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose determinant is $\delta = ad - bc$ and whose trace is $\tau = a + d$. Then the eigenvalues $\omega_{1,2}$ of A can be expressed in terms of δ and τ as follows:

If

$$\tau^2 - 4\delta \geq 0, \tag{30}$$

then one has

$$\omega_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\delta}\right), \tag{31}$$

and if

$$\tau^2 - 4\delta < 0, \tag{32}$$

then one has

$$\omega_{1,2} = \frac{1}{2}\left(\tau \pm i\sqrt{4\delta - \tau^2}\right). \tag{33}$$

Thus one has the following theorems:

Theorem 8 *The fixed points of the map (1) with the Jacobian matrix A are not all asymptotically stable, if one or more than one of the following conditions holds:*

$$\tau^2 - 4\delta \geq 0, |\tau| > 2 \tag{34}$$

$$\tau^2 - 4\delta \geq 0, \delta + 1 < \tau < 2, \delta < 1 \tag{35}$$

$$\tau^2 - 4\delta \geq 0, -(\delta + 1) < \tau < -2, \delta > 1 \tag{36}$$

$$\tau^2 - 4\delta \geq 0, 2 < \tau < \delta + 1, \delta > 1 \tag{37}$$

$$\tau^2 - 4\delta \geq 0, -2 < \tau < -(\delta + 1), \delta < 1 \tag{38}$$

$$|\tau| < 2\sqrt{\delta}, \delta > 1 \tag{39}$$

Theorem 9 *The fixed points of the map (1) with the Jacobian matrix A are all asymptotically stable if one of the following conditions holds:*

$$\begin{cases} \tau - 1 < \delta < \frac{\tau^2}{4} \\ 0 \leq \tau < 2 \end{cases} \tag{40}$$

or

$$\begin{cases} -\tau - 1, < \delta < \frac{\tau^2}{4} \\ -2 < \tau \leq 0 \end{cases} \tag{41}$$

or

$$0 < \delta < 1, |\tau| < 2\sqrt{\delta} \tag{42}$$

For the map (1), one has

$$\begin{cases} \delta = \xi_1 x^2 + \xi_2 y^2 + \xi_3 xy + \xi_4 x + \xi_5 y + \xi_6 \\ \tau = \xi_7 x + \xi_8 y + \xi_9, \end{cases} \tag{43}$$

where $(\xi_i)_{1 \leq i \leq 6}$ are given in (17) and

$$\begin{cases} \xi_7 = 2a_3 + b_5 \\ \xi_8 = a_5 + 2b_4 \\ \xi_9 = a_1 + b_2. \end{cases} \tag{44}$$

6.1 A zone of possible chaotic orbits

We remark from some special cases of the map (1) that a condition for the existence of chaos is that the map has at least one saddle or unstable fixed point, and thus one has the following theorem:

Theorem 10 *A possible range for chaos in the 2-D quadratic map (1) is the set* $\Omega_1 = \cup_{i=5}^{i=10} C_i \subset \mathbb{R}^{12}$, where $(C_i)_{5\leq i\leq 10}$ are the following subsets of $\mathbb{R}^{12}$:

$$C_5 : \begin{cases} \xi_{10} > 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ b_3\xi_8 > 0, \xi_{16} < 0, \xi_{17}^2 - 4\xi_{16}\xi_{18} < 0, \text{ or} \\ b_3\xi_8 < 0, \xi_{19} < 0, \xi_{20}^2 - 4\xi_{19}\xi_{21} < 0 \end{cases} \quad (45)$$

$$C_6 : \begin{cases} \xi_{10} > 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ \xi_1 < 0, \xi_{22} < 0, \xi_{23}^2 - 4\xi_{22}\xi_{24} < 0 \\ b_3\xi_8 < 0, \xi_{16} < 0, \xi_{17}^2 - 4\xi_{16}\xi_{18} < 0 \\ \xi_1 < 0, \xi_{25} < 0, \xi_{26}^2 - 4\xi_{25}\xi_{27} \end{cases} \quad (46)$$

$$C_7 : \begin{cases} \xi_{10} > 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ \xi_1 > 0, \xi_{28} < 0, \xi_{29}^2 - 4\xi_{28}\xi_{30} < 0 \\ b_3\xi_8 < 0, \xi_{19} < 0, \xi_{20}^2 - 4\xi_{19}\xi_{21} < 0 \\ \xi_1 > 0, \xi_{25} < 0, \xi_{26}^2 - 4\xi_{25}\xi_{27} \end{cases} \quad (47)$$

$$C_8 : \begin{cases} \xi_{10} > 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ b_3\xi_8 > 0, \xi_{16} < 0, \xi_{17}^2 - 4\xi_{16}\xi_{18} < 0 \\ \xi_1 > 0, \xi_{22} < 0, \xi_{23}^2 - 4\xi_{22}\xi_{24} < 0 \\ \xi_1 > 0, \xi_{25} < 0, \xi_{26}^2 - 4\xi_{25}\xi_{27} \end{cases} \quad (48)$$

$$C_9 : \begin{cases} \xi_{10} > 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ b_3\xi_8 > 0, \xi_{19} < 0, \xi_{20}^2 - 4\xi_{19}\xi_{21} < 0 \\ \xi_1 < 0, \xi_{28} < 0, \xi_{29}^2 - 4\xi_{28}\xi_{30} < 0 \\ \xi_1 < 0, \xi_{25} < 0, \xi_{26}^2 - 4\xi_{25}\xi_{27} \end{cases} \quad (49)$$

$$C_{10}: \begin{cases} \xi_{10} < 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ \xi_1 > 0, \xi_{25} < 0, \xi_{26}^2 - 4\xi_{25}\xi_{27}, \end{cases} \tag{50}$$

where

$$\begin{cases} \xi_{10} = \xi_7^2 - 4\xi_1 \\ \xi_{11} = \xi_8^2 - 4\xi_2 \\ \xi_{12} = 2\xi_7\xi_8 - 4\xi_3 \\ \xi_{13} = 2\xi_7\xi_9 - 4\xi_4 \\ \xi_{14} = 2\xi_8\xi_9 - 4\xi_5 \\ \xi_{15} = \xi_9^2 - 4\xi_6 \\ \xi_{16} = b_5^2\xi_8^2 - 4b_3b_4\xi_8^2 \\ \xi_{17} = 2b_5\xi_7\xi_8 + 2b_1b_5\xi_8^2 - 4b_2b_3\xi_8^2\xi_9 \\ \xi_{18} = 8b_3\xi_8 + 2b_1\xi_7\xi_8 + \xi_7^2 - 4b_0b_3\xi_8^2 + b_1^2\xi_8^2 \\ \xi_{19} = b_5^2\xi_8^2 - 4b_3b_4\xi_8^2 \\ \xi_{20} = 2b_5\xi_7\xi_8 - 4b_2b_3\xi_8^2 + 2b_1b_5\xi_8^2 \end{cases} \tag{51}$$

and

$$\begin{cases} \xi_{21} = 2b_1\xi_7\xi_8 - 8b_3\xi_8 - 4b_3\xi_8\xi_9 + \xi_7^2 - 4b_0b_3\xi_8^2 + b_1^2\xi_8^2 \\ \xi_{22} = \xi_3^2 - 4\xi_1\xi_2 \\ \xi_{23} = 4\xi_1\xi_8 - 2\xi_3\xi_7 - 4\xi_1\xi_5 + 2\xi_3\xi_4 \\ \xi_{24} = 4\xi_1\xi_9 - 4\xi_1\xi_6 - 4\xi_1 - 2\xi_4\xi_7 + \xi_4^2 + \xi_7^2 \\ \xi_{25} = \xi_3^2 - 4\xi_1\xi_2 \\ \xi_{26} = 2\xi_3\xi_4 - 4\xi_1\xi_5 \\ \xi_{27} = 4\xi_1 - 4\xi_1\xi_6 + \xi_4^2 \\ \xi_{28} = \xi_3^2 - 4\xi_1\xi_2 \\ \xi_{29} = 2\xi_3\xi_4 - 4\xi_1\xi_8 + 2\xi_3\xi_7 - 4\xi_1\xi_5 \\ \xi_{30} = 2\xi_4\xi_7 - 4\xi_1\xi_6 - 4\xi_1\xi_9 - 4\xi_1 + \xi_4^2 + \xi_7^2. \end{cases} \tag{52}$$

The proof is given by Theorem 4 as follows:

Proof. From (43), one has that $\tau^2 - 4\delta = \xi_{10}x^2 + (\xi_{12}y + \xi_{13})x + \xi_{11}y^2 + \xi_{14}y + \xi_{15} \geq 0$ if and only if $\xi_{10} > 0$ and $\left(\xi_{12}^2 - 4\xi_{10}\xi_{11}\right)y^2 + (2\xi_{12}\xi_{13} - 4\xi_{10}\xi_{14})y - 4\xi_{10}\xi_{15} + \xi_{13}^2 < 0$ for all $y \in \mathbb{R}$, i.e., $\xi_{10} > 0$, $\xi_{12}^2 - 4\xi_{10}\xi_{11} < 0$ and $\xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2}$. On the other hand, we have $|\tau| = |\xi_7 x + \xi_8 y + \xi_9| > 2$ if and only if $\xi_7 x + \xi_8 y + \xi_9 > 2$ or $\xi_7 x + \xi_8 y + \xi_9 < -2$ using the curve C_0 given in (20). Then one has $b_3\xi_8 x^2 + (b_5\xi_8 y + \xi_7 + b_1\xi_8)x + b_4\xi_8 y^2 + b_2\xi_8 y\xi_9 + b_0\xi_8 - 2 > 0$ if $b_3\xi_8 > 0, \xi_{16} < 0, \xi_{17}^2 - 4\xi_{16}\xi_{18} < 0$, or $b_3\xi_8 x^2 + (\xi_7 + b_1\xi_8 + b_5\xi_8 y)x + b_4\xi_8 y^2 + b_2\xi_8 y + b_0\xi_8 + \xi_9 + 2 < 0$ if $b_3\xi_8 < 0, \xi_{19} < 0, \xi_{20}^2 - 4\xi_{19}\xi_{21} < 0$. The other cases given in (45) to (50) can be obtained from the conditions (34) to (39) using the same logic. ∎

Theorem 11 *The set $\Omega_1 = \cup_{i=5}^{i=10} C_i \subset \mathbb{R}^{12}$ is not empty.*

Proof. The Hénon map [2] has a chaotic attractor with a saddle fixed point for $a_0 = b_1 = 1, a_2 = 0.3, a_3 = -1.4$, and $a_1 = a_4 = a_5 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$. ∎

6.2 Zones of stable fixed points

Generally, the geometric structure of an orbit of the map (1) depends on the number of fixed points. It can be verified that map (1) has at most 4 fixed points. Then if all its fixed points are stable (the absolute value of their eigenvalues does not exceed 1), the map (1) converges to a fixed point, i.e., for the following subsets of $\mathbb{R}^{12}$ defined by:

$$C_{11}: \begin{cases} \xi_1 < 0, \xi_{28} < 0, \xi_{29}^2 - 4\xi_{28}\xi_{30} < 0 \\ \xi_{10} > 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ b_3\xi_8 > 0, \xi_{31} < 0, \xi_{32}^2 - 4\xi_{31}\xi_{33} < 0 \\ b_3\xi_8 < 0, \xi_{16} < 0, \xi_{17}^2 - 4\xi_{16}\xi_{18} < 0 \end{cases} \tag{53}$$

or

$$C_{12}: \begin{cases} b_3\xi_8 > 0, \xi_{16} < 0, \xi_{17}^2 - 4\xi_{16}\xi_{18} < 0 \\ \xi_{10} > 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \\ b_3\xi_8 > 0, \xi_{19} < 0, \xi_{20}^2 - 4\xi_{19}\xi_{21} < 0 \\ b_3\xi_8 < 0, \xi_{31} < 0, \xi_{32}^2 - 4\xi_{31}\xi_{33} < 0 \end{cases} \tag{54}$$

or

$$C_{13}: \begin{cases} \xi_1 > 0, \xi_{36} < 0, \xi_{37}^2 - 4\xi_{36}\xi_{38} < 0 \\ \xi_1 < 0, \xi_{25} < 0, \xi_{26}^2 - 4\xi_{25}\xi_{27} \\ \xi_{10} < 0, \xi_{12}^2 - 4\xi_{10}\xi_{11} < 0, \xi_{15} > \frac{\xi_{10}\xi_{14}^2 - \xi_{12}\xi_{13}\xi_{14} + \xi_{11}\xi_{13}^2}{4\xi_{10}\xi_{11} - \xi_{12}^2} \end{cases} \tag{55}$$

where

$$\begin{cases} \xi_{31} = b_5^2\xi_8^2 - 4b_3b_4\xi_8^2 \\ \xi_{32} = 2b_5\xi_7\xi_8 + \xi_7^2 - 4b_2b_3\xi_8^2 + 2b_1b_5\xi_8^2 \\ \xi_{33} = 2b_1\xi_7\xi_8 - 4b_3\xi_8\xi_9 - 4b_0b_3\xi_8^2 + b_1^2\xi_8^2 \\ \xi_{35} = \xi_3^2 - 4\xi_1\xi_2 \\ \xi_{36} = 2\xi_3\xi_4 - 4\xi_1\xi_5 \\ \xi_{37} = -4\xi_1\xi_6 + \xi_4^2. \end{cases} \tag{56}$$

Hence the following theorem is proved:

Theorem 12 *If* $(a_i, b_i)_{1\leq i\leq 6} \in \Omega_2 = \cup_{i=11}^{i=13} C_i$*, then the 2-D quadratic map (1) converges to a fixed point.*

Theorem 13 *The set* $\Omega_2 = \cup_{i=11}^{i=13} C_i \subset \mathbb{R}^{12}$ *is not empty.*

Proof. The delayed logistic map [9] is asymptotically stable for all $-7 \leq a_5 \leq 7$ and $-6 \leq b_1 \leq 6$, with $a_2 = -a_5$ and $a_0 = a_1 = a_3 = a_4 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$. ■

The above analysis is not true in some cases where at least one fixed point is not asymptotically stable. For example, in [17], there are some regions in a_4-a_1 space where two coexisting attractors occur as shown in the black region of Fig. 2. For example, with $a_4 = 1$ and $a_1 = -0.8$, a fixed point (at $x = y = 0.4329311$) coexists with a period-3 orbit, and with $a_4 = 1$ and $a_1 = -0.8$, a fixed point (at $x = y = 0.445362$) coexists with a quasi-periodic orbit. Another example can be found in [15].

7 Classification of chaotic orbits of the 2-D quadratic map according to their number of nonlinearities

In this section, all possible cases of chaos in the general 2-D quadratic map (1) are classified according to their number of nonlinearities, and for each class,

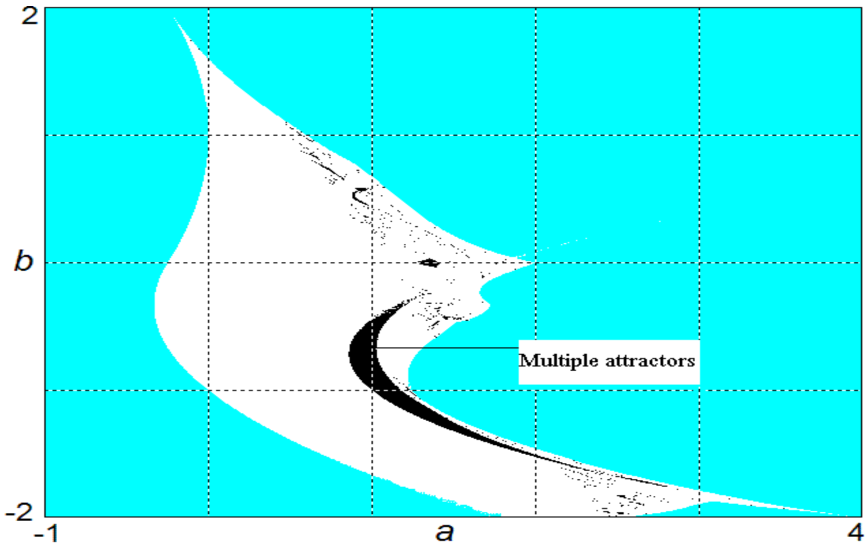

Figure 2: The regions of a_4-a_1 space for multiple attractors.

several examples are given showing the structure of the parameter space for the considered map with selected values of the vectors $(a_i, b_i)_{1\leq i\leq 6} \subset \mathbb{R}^{12}$. In order to simplify the search for the possible classes of chaos in the map (1), we introduce the following notations: A special case of the map (1) is of type $(m-l)$ if it has m nonlinearities, and l indicates the number of a possible cases of it. This classification is based primarily on the above theorems of equivalence between quadratic maps. As a test of the above theorems, we give detailed examples for cases with one nonlinearity including the Hénon map [2] and the map given in [17]. The other cases can be studied by the same logic.

7.1 2-D quadratic maps with one nonlinearity

For the class of one nonlinearity, we have three cases:

$$(1-1)\quad \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k \end{cases}$$
$$(1-2)\quad \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_4 y_k^2 \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k \end{cases} \tag{57}$$
$$(1-3)\quad \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_5 x_k y_k \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k. \end{cases}$$

The maps $(1-1)$, including the Hénon map [2] obtained for $a_1 = 0, b_0 = 0$, and $b_2 = 0$, are shown in Fig. 1(a). The maps $(1-2)$, including the map studied in [17] obtained for $a_2 = 0, b_0 = 0$, and $b_2 = 0$, are shown in Fig. 1(b). The maps $(1-3)$, including the delayed logistic map [9] obtained for $a_0 = 0, a_2 = 0, b_0 = 0, b_1 = 1$, and $b_2 = 0$, and the two cases obtained for $a_0 = 1, a_1 = 0, b_0 = 0, b_1 = 1$, and $b_2 = 0$, or $a_0 = 1, a_2 = 0, b_0 = 0, b_1 = 1$, and$b_2 = 0$, where it is shown in [18] that both maps are not topologically equivalent to the delayed logistic map, [9] the Hénon map, [2] or the map given in [17]. Some simple cases of the maps $(1-i)_{i=1,3}$ that give chaotic attractors are as follows:

$$\begin{array}{ll} (1-1) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - a_4 x_k^2 + a_2 y_k, x_k) \\ (1-2) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - a_4 y_k^2 + a_1 x_k, x_k) \\ (1-3) & \begin{cases} (x_{k+1}, y_{k+1}) \longrightarrow (a_1 x_k - a_1 x_k y_k, x_k) \\ (x_{k+1}, y_{k+1}) \longrightarrow (1 - a_5 x_k y_k + a_2 y_k, x_k) \\ (x_{k+1}, y_{k+1}) \longrightarrow (1 - a_5 x_k y_k + a_1 x_k, x_k). \end{cases} \end{array} \tag{58}$$

As an example, we apply the above theorems to the map $(1-1)$ as follows: The map $(1-1)$ has no fixed points if $(a_i, b_i) \in \Omega \subset \mathbb{R}^{12}$, where Ω is given by (59) or (60) below:

$$\Omega : \begin{cases} b_2 \neq 1 \\ \\ \left(a_1 - 1 - \frac{a_2}{b_2 - 1}\right)^2 < 4a_3 \left(-\frac{b_0 a_2}{b_2 - 1} + a_0\right) \\ \\ a_3 \left(-\frac{b_0 a_2}{b_2 - 1} + 1\right) > 0 \end{cases} \tag{59}$$

or

$$\Omega:\begin{cases} b_2=1, a_2=0 \\ b_0-a_1b_0^3a_3+1\neq 0. \end{cases} \tag{60}$$

On the other hand, the map $(1-1)$ has one fixed point:

$$P=\left(\frac{2\left(b_0a_2-b_2+1\right)}{a_1b_2-a_2-b_2-a_1+1},\frac{-\left(b_0-2b_2-a_1b_0+b_0a_2-b_0b_2+a_1b_0b_2+2\right)}{\left(a_1b_2-a_2-b_2-a_1+1\right)\left(b_2-1\right)}\right) \tag{61}$$

if $(a_i,b_i)\in\bar{\Omega}\subset\mathbb{R}^{12}$, where

$$\bar{\Omega}:b_2\neq 1, a_1\neq\frac{a_2+b_2-1}{\left(b_2-1\right)}, b_2\neq b_0a_2+1, a_3=\frac{-\left(a_1b_2-a_2-b_2-a_1+1\right)^2}{4\left(b_0a_2-b_2+1\right)\left(b_2-1\right)} \tag{62}$$

and has two fixed points $\left(x_1,\frac{-x_1-b_0}{b_2-1}\right)$ and $\left(x_2,\frac{-x_2-b_0}{b_2-1}\right)$, where

$$\begin{cases} x_1=\frac{-\left(a_1-1-\frac{a_2}{b_2-1}\right)+\sqrt{\left(a_1-1-\frac{a_2}{b_2-1}\right)^2-4a_3\left(-b_0\frac{a_2}{b_2-1}+1\right)}}{2a_3} \\ x_2=\frac{-\left(a_1-1-\frac{a_2}{b_2-1}\right)-\sqrt{\left(a_1-1-\frac{a_2}{b_2-1}\right)^2-4a_3\left(-b_0\frac{a_2}{b_2-1}+1\right)}}{2a_3} \end{cases} \tag{63}$$

if $(a_i,b_i)\in\bar{\Omega}\subset\mathbb{R}^{12}$, where

$$\bar{\Omega}:\begin{cases} b_2\neq 1 \\ \frac{\left(a_1+a_2+b_2-a_1b_2-1\right)^2}{4}>a_3\left(b_2-1\right)\left(b_2-b_0a_2-1\right). \end{cases} \tag{64}$$

We remark that the chaotic Hénon attractor has at least one saddle fixed point. We then use this idea to identify some possible chaotic systems for the map $(1-1)$. The Jacobian matrix of the map $(1-1)$ is given by

$$J\left(x,y\right)=\begin{pmatrix} a_1+2a_3x & a_2 \\ 1 & b_2 \end{pmatrix}. \tag{65}$$

From the conditions (25) to (28) the map $(1-1)$ has one fixed point, and it is of the saddle-type if $(a_i,b_i)_{0\leq i\leq 5}\in\Omega_1$, where Ω_1 is the union of the following conditions:

$$\left|a_2-2b_2+b_2^2+1\right|>-\operatorname{sgn}\left(b_2-1\right)\left(a_2-2b_2+b_2^2+1\right) \tag{66}$$

$$\begin{cases} |a_2 - 2b_2 + b_2^2 + 1| < -\operatorname{sgn}(b_2 - 1)(a_2 + 2b_2 + b_2^2 - 3) \\ -\operatorname{sgn}(b_2 - 1)(a_2 + 2b_2 + b_2^2 - 3) > 0 \end{cases} \tag{67}$$

$$\begin{cases} |a_2 - 2b_2 + b_2^2 + 1| < \operatorname{sgn}(b_2 - 1)(a_2 - 2b_2 + b_2^2 + 1) \\ \operatorname{sgn}(b_2 - 1)(a_2 - 2b_2 + b_2^2 + 1) > 0 \end{cases} \tag{68}$$

$$\left|a_2 - 2b_2 + b_2^2 + 1\right| > \operatorname{sgn}(b_2 - 1)\left(a_2 + 2b_2 + b_2^2 - 3\right), \tag{69}$$

where sgn(.) is the standard signum function that gives the sign of its argument. On the other hand, the map $(1-1)$ has two fixed points, one of which is a saddle point if $(a_i, b_i)_{0 \le i \le 5} \in \Omega_1$, where Ω_1 is the union of the following conditions:

$$\begin{cases} 4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2 \ge 0 \\ -2 + a_1 + b_2 + 2a_3x_1 > 0 \\ (2a_3b_2 - 2a_3)x_1 + a_1b_2 - a_2 - b_2 - a_1 + 1 > 0 \end{cases} \tag{70}$$

$$\begin{cases} 4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2 \ge 0 \\ \sqrt{4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2} > 2 + a_1 + b_2 + 2a_3x_1 \end{cases} \tag{71}$$

$$\begin{cases} 4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2 \ge 0 \\ \sqrt{4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2} > 2 - b_2 - 2a_3x_1 - a_1 \end{cases} \tag{72}$$

$$\begin{cases} 4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2 \ge 0 \\ \sqrt{4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2} < -2 - a_1 - b_2 - 2a_3x_1 \\ -2 - a_1 - b_2 - 2a_3x_1 > 0 \end{cases} \tag{73}$$

$$\begin{cases} 4a_3^2x_1^2 + (4a_1a_3 - 4a_3b_2)\,x_1 + 4a_2 - 2a_1b_2 + a_1^2 + b_2^2 < 0 \\ 0 < b_2 - a_2 + a_2\frac{b_2}{b_2-1} + b_2\sqrt{\left(a_1 - 1 - \frac{a_2}{b_2-1}\right)^2 - 4a_3\left(-b_0\frac{a_2}{b_2-1} + 1\right)} < 1. \end{cases} \tag{74}$$

A description of the dynamics of the Hénon map is given in [2-3-16].

For the second map $(1-1)$, we consider the map studied in [17] given by

$$\begin{cases} x_{k+1} = 1 + a_1x_k + a_4y_k^2 \\ y_{k+1} = x_k. \end{cases} \tag{75}$$

Because it is proved in [17] that one of the fixed points of the map (75) is either unstable or of a saddle type, one has $\Omega_1 = \bar{\Omega}$. The sets $\Omega, \bar{\Omega}$ (A, B in Fig. 3) $\subset \mathbb{R}^{12}$ are defined by:

$$\Omega : a_4 < -\left(\frac{-a_1+1}{2}\right)^2 \tag{76}$$

$$\bar{\Omega} = \Omega_1 : a_4 \geq -\left(\frac{-a_1+1}{2}\right)^2. \tag{77}$$

A schematic representation of these results is given in Fig. 3, where C is the line $C : a_4 = -\left(\frac{-a_1+1}{2}\right)^2$.

The third case is studied in [18].

7.2 2-D quadratic maps with two nonlinearities

For the case of two nonlinearities, we have six cases given by

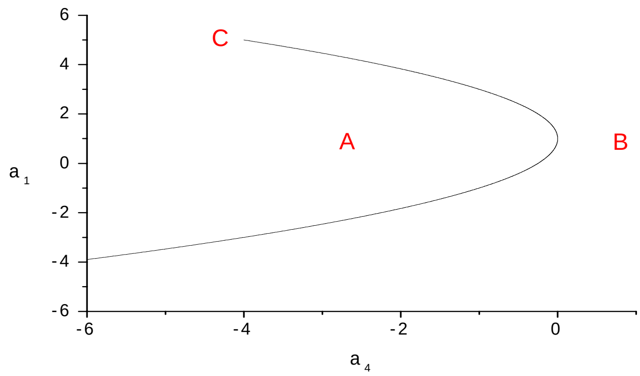

Figure 3: Regions in the a_4-a_1 plane for unbounded and bounded orbits for the map (75).

$$
\begin{array}{ll}
(2-1) & \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k \end{cases} \\
(2-2) & \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_5 x_k y_k \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k \end{cases} \\
(2-3) & \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 \end{cases} \\
(2-4) & \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_4 y_k^2 \end{cases} \\
(2-5) & \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_5 x_k y_k \end{cases} \\
(2-6) & \begin{cases} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_5 x_k y_k \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_5 x_k y_k. \end{cases}
\end{array}
\tag{78}
$$

Note that a special case of the map $(2-1)$ is under preparation in [19].

Some simple cases of the maps $(2-i)_{i=1,6}$ that give chaotic attractors are given in the following table:

$$\begin{array}{lll} (2-1) & (x_{k+1}, y_{k+1}) \longrightarrow (1-1.4x_k^2 - 1.7y_k^2, x_k) \\ (2-2) & (x_{k+1}, y_{k+1}) \longrightarrow (1-1.6x_k^2 + 0.7x_k y_k, x_k) \\ (2-3) & (x_{k+1}, y_{k+1}) \longrightarrow (1+0.2y_k - 1.7x_k^2, 0.7x_k^2) \\ (2-4) & (x_{k+1}, y_{k+1}) \longrightarrow (1+0.1y_k - 1.1x_k^2, x_k - y_k^2) \\ (2-5) & (x_{k+1}, y_{k+1}) \longrightarrow (1-0.8y_k - 1.3x_k^2, 1-1.6x_k y_k) \\ (2-6) & (x_{k+1}, y_{k+1}) \longrightarrow (1+0.3x_k - 1.5x_k y_k, 1-1.9x_k y_k). \end{array} \tag{79}$$

The corresponding chaotic attractors are shown in Fig. 4.

7.3 2-D quadratic maps with three nonlinearities

For the case of three nonlinearities, we have 10 cases:

$$\left\{ \begin{array}{c} (3-1): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k \end{array} \right. \\ (3-2): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 \end{array} \right. \\ (3-3): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_4 y_k^2 \end{array} \right. \\ (3-4): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_5 x_k y_k \end{array} \right. \\ (3-5): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 \end{array} \right. \\ (3-6): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_4 y_k^2 \end{array} \right. \\ (3-7): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_5 x_k y_k \end{array} \right. \\ (3-8): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_4 y_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 \end{array} \right. \\ (3-9): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_4 y_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_4 y_k^2 \end{array} \right. \\ (3-10): \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_4 y_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_5 x_k y_k. \end{array} \right. \end{array} \right. \tag{80}$$

Some simple cases of the maps $(3-i)_{i=1,10}$ that give chaotic attractors are listed in the following table:

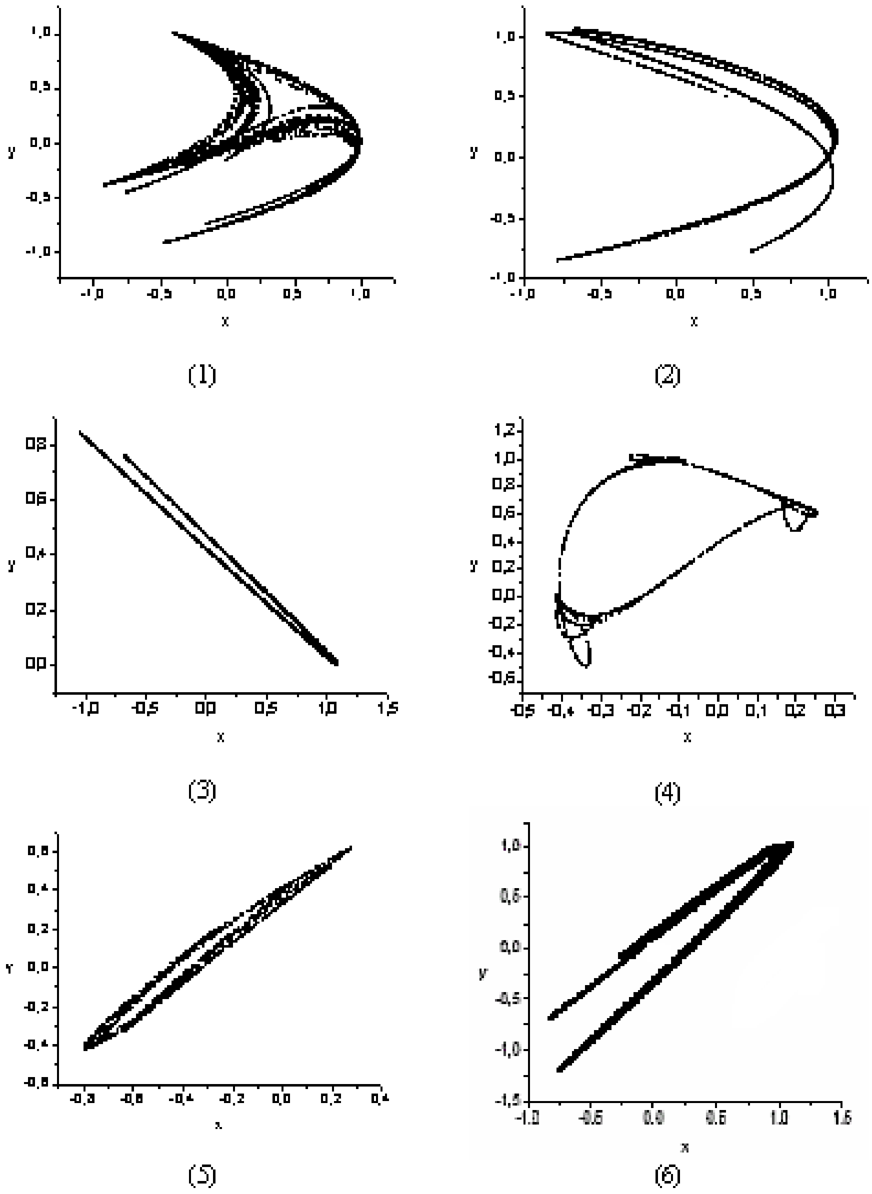

Figure 4: Chaotic attractors of the type $(2-i)_{1\leq i\leq 6}$.

$$
\begin{array}{ll}
(3-1) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.4x_k^2 - 1.5y_k^2 - 1.8x_k y_k, x_k) \\
(3-2) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.4x_k^2 - 1.7y_k^2, -y_k + 0.6x_k^2) \\
(3-3) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.4x_k^2 - 1.7y_k^2, x_k + 0.9y_k^2) \\
(3-4) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.4x_k^2 - 1.1y_k^2, x_k + 0.9x_k y_k) \\
(3-5) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.3x_k^2 - 1.1x_k y_k, -0.3y_k - x_k^2) \\
(3-6) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.3x_k^2 - x_k y_k, x_k - y_k^2) \\
(3-7) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.5x_k^2 + 0.6x_k y_k, 1 - 1.2x_k y_k) \\
(3-8) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.2y_k^2 + 0.6x_k y_k, 0.4y_k - x_k^2) \\
(3-9) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.5y_k^2 - 0.9x_k y_k, x_k - 0.6y_k^2) \\
(3-10) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.4y_k^2 + 0.4x_k y_k, 1 + 0.1x_k y_k)\,.
\end{array}
\tag{81}
$$

The corresponding chaotic attractors are shown in Figs. 5 and 6.

7.4 2-D quadratic maps with four nonlinearities

For the case of four nonlinearities, we have 7 cases:

$$
\begin{array}{ll}
(4-1) & \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 \end{array} \right. \\
(4-2) & \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_5 x_k y_k \end{array} \right. \\
(4-3) & \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 + b_4 y_k^2 \end{array} \right. \\
(4-4) & \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 + b_5 x_k y_k \end{array} \right. \\
(4-5) & \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_4 y_k^2 + b_5 x_k y_k \end{array} \right. \\
(4-6) & \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 + b_5 x_k y_k \end{array} \right. \\
(4-7) & \left\{ \begin{array}{c} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_5 x_k y_k \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_4 y_k^2 + b_5 x_k y_k. \end{array} \right.
\end{array}
\tag{82}
$$

Some simple cases of the maps $(4-i)_{i=1,7}$ that give chaotic attractors are listed in the following table:

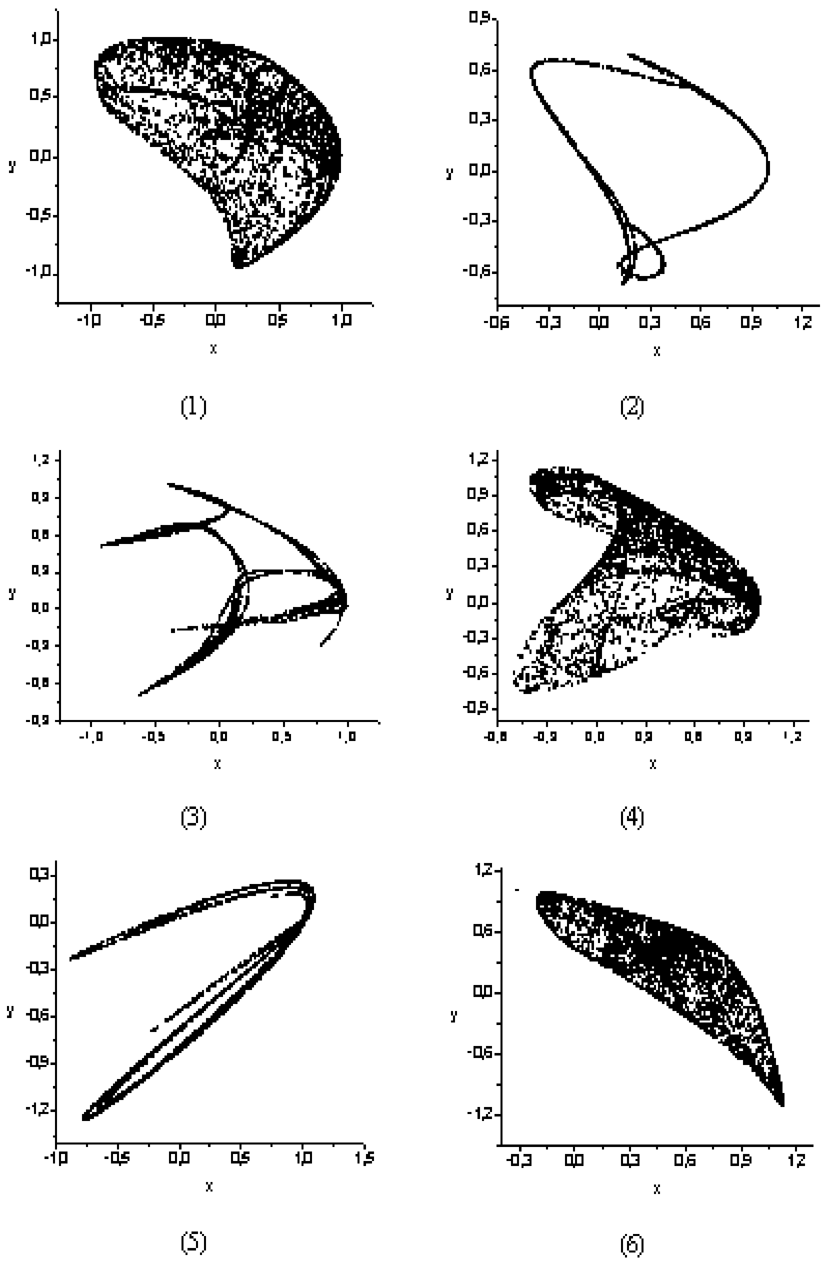

Figure 5: Chaotic attractors of the type $(3-i)_{1\leq i\leq 6}$.

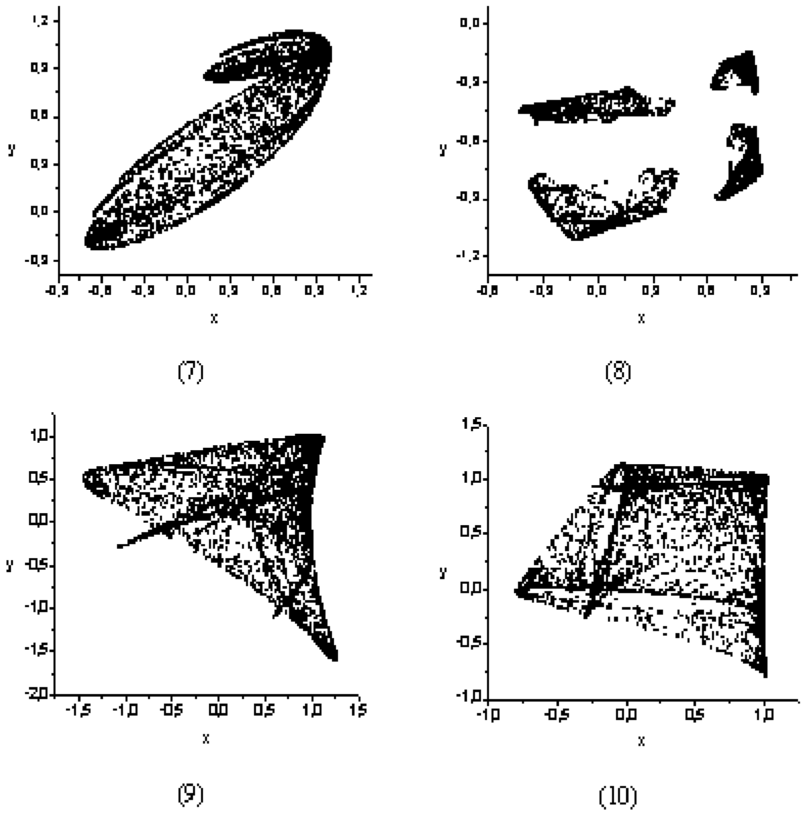

Figure 6: Chaotic attractors of the type $(3-i)_{7\leq i\leq 10}$.

$$
\begin{array}{lll}
(4-1) & (x_{k+1}, y_{k+1}) \longrightarrow (1 + 0.1x_k^2 - 0.5y_k^2 - 0.4x_k y_k, -1.6x_k^2) \\
(4-2) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.3x_k^2 - 1.5y_k^2 - 1.2x_k y_k, 1 + 0.6x_k y_k) \\
(4-3) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.3x_k^2 - 0.6y_k^2, 1 + x_k^2 - 1.4y_k^2) \\
(4-4) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.3x_k^2 - 0.8y_k^2, 1 - x_k^2 + 1.3x_k y_k) \\
(4-5) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.4x_k^2 - 0.7y_k^2, 1 - y_k^2 + 1.3x_k y_k) \\
(4-6) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.4x_k^2 - 1.8x_k y_k, 0.6x_k^2 - 1.8x_k y_k) \\
(4-7) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.3x_k^2 - 0.7x_k y_k, 1 - 1.1y_k^2 + 1.3x_k y_k) \,.
\end{array}
\tag{83}
$$

The corresponding chaotic attractors are shown in Fig. 7.

7.5 2-D quadratic maps with five nonlinearities

For the case of five nonlinearities, we have 3 cases:

$$
\begin{array}{ll}
(5-1) & \left\{ \begin{array}{l} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 + a_5 x_k y_k \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 + b_4 y_k^2 \end{array} \right. \\
(5-2) & \left\{ \begin{array}{l} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 + a_5 x_k y_k \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_3 x_k^2 + b_5 x_k y_k \end{array} \right. \\
(5-3) & \left\{ \begin{array}{l} x_{k+1} = a_0 + a_1 x_k + a_2 y_k + a_3 x_k^2 + a_4 y_k^2 + a_5 x_k y_k \\ \\ y_{k+1} = b_0 + b_1 x_k + b_2 y_k + b_4 y_k^2 + b_5 x_k y_k. \end{array} \right.
\end{array}
\tag{84}
$$

Some simple cases of the maps $(4-i)_{i=1,7}$ that give chaotic attractors are listed in the following table:

$$
\begin{array}{lll}
(5-1) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 1.8x_k^2 + 1.4y_k^2 + 0.1x_k y_k, -0.2x_k^2 + 1.8y_k^2) \\
(5-2) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.5x_k^2 - y_k^2 + x_k y_k, -1.2x_k^2 - x_k y_k) \\
(5-3) & (x_{k+1}, y_{k+1}) \longrightarrow (1 - 0.5x_k^2 - 1.7y_k^2 + 0.9x_k y_k 1 - 0.4y_k^2 - 1.5x_k y_k) \,.
\end{array}
\tag{85}
$$

The corresponding chaotic attractors are shown in Fig. 8.

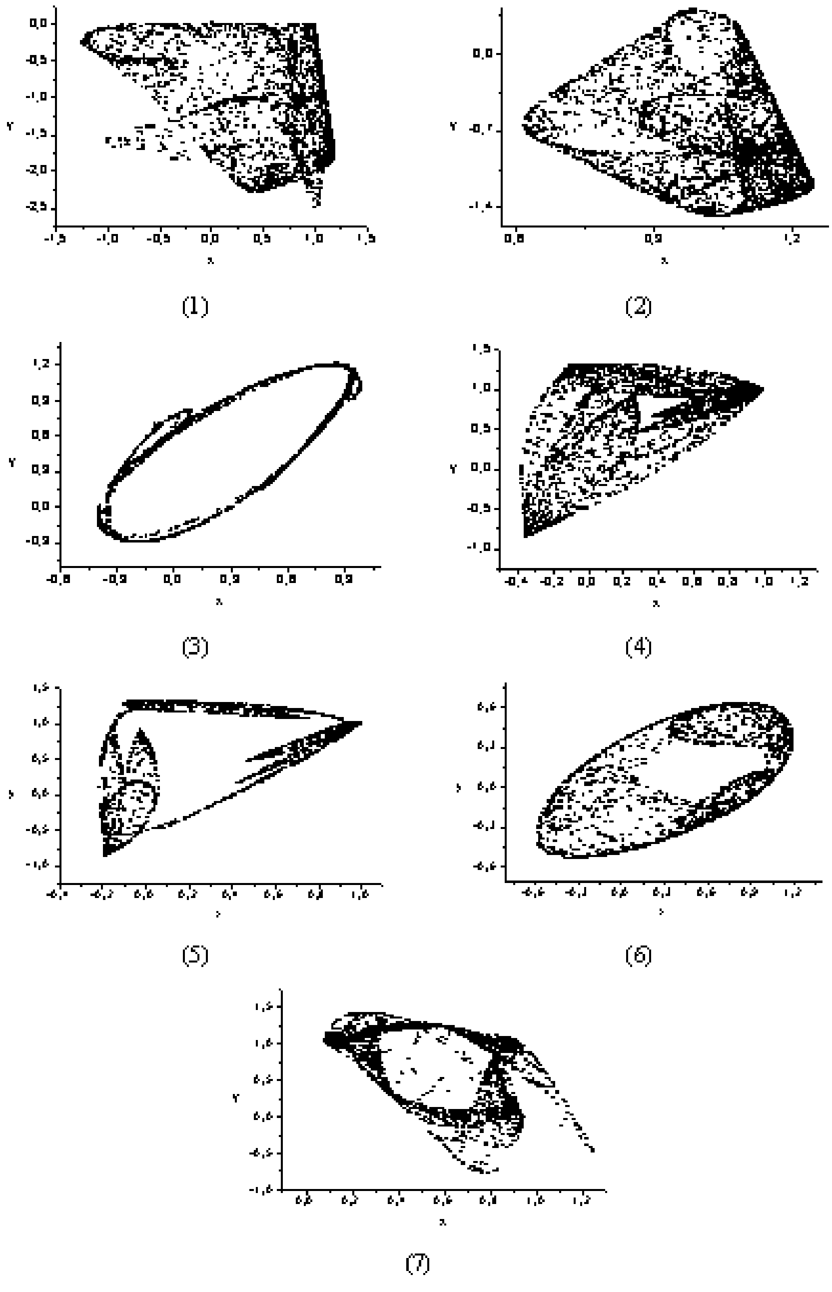

Figure 7: Chaotic attractors of the type $(4-i)_{1\leq i\leq 7}$.

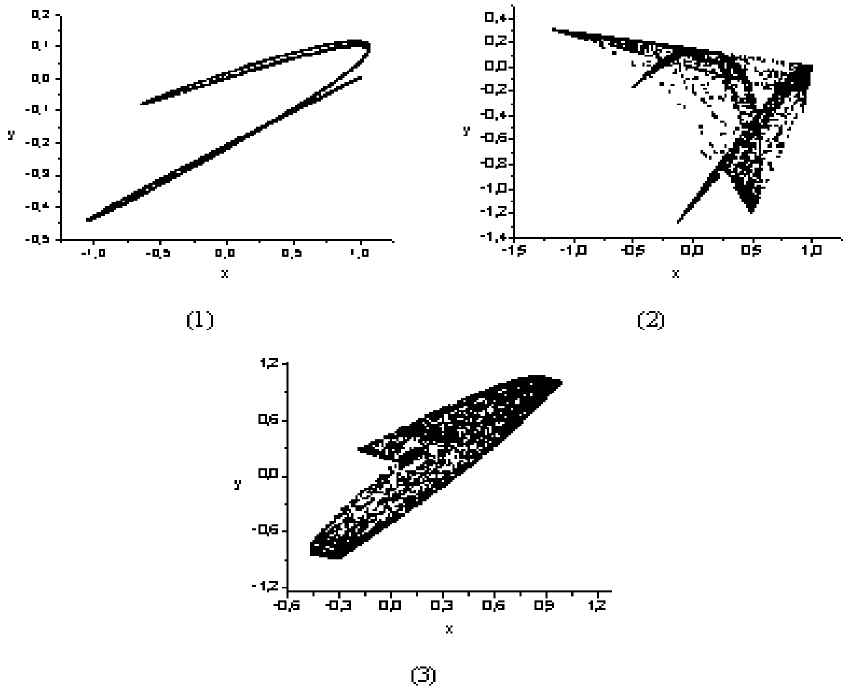

Figure 8: Chaotic attractors of the type $(5-i)_{1\leq i\leq 3}$.

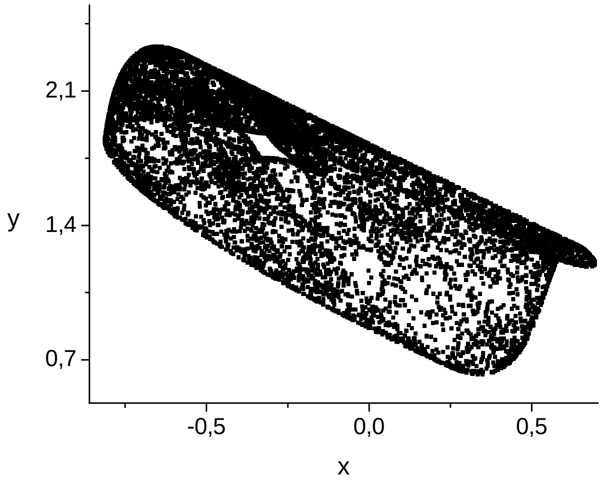

Figure 9: Chaotic attractor of the type $(6-1)$.

7.6 2-D quadratic maps with six nonlinearities

For the case of six nonlinearities, we have only one case given by the 2-D quadratic map (1) itself. A simple example of it is the following:

$$\begin{cases} x_{k+1} = 1 + 0.1x_k^2 - 0.5y_k^2 - 0.6x_k y_k \\ y_{k+1} = 1 - x_k^2 + 0.4y_k^2 + 0.7x_k y_k. \end{cases} \tag{86}$$

The corresponding chaotic attractor is shown in Fig. 9.

8 Numerical analysis

In this section, we investigate some important dynamical behaviors observed for some special cases of the map (1).

For the case of 2-D quadratic maps with one nonlinearity, it is well known that the Hénon map [2] typically undergoes a period-doubling route to chaos when the parameters are varied as shown in Fig. 11. Furthermore, the minimal quadratic chaotic attractors considered in [17-18] result from a quasi-periodic route to chaos as shown in Figs. 10, 12, 13, and 14, where the regions

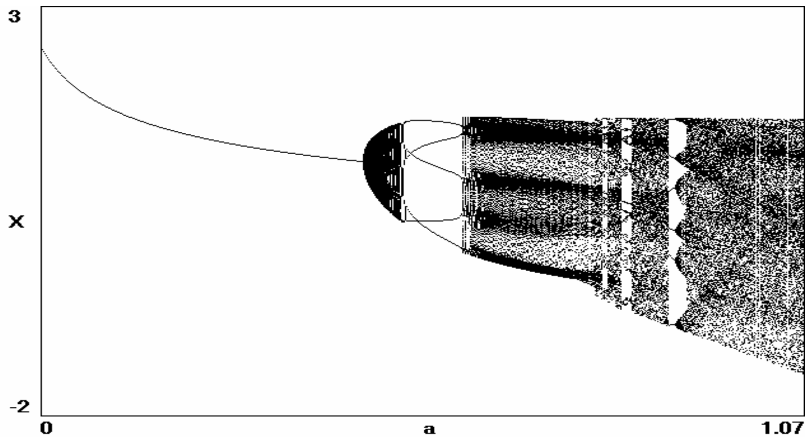

Figure 10: The quasi-periodic route to chaos for the map given in [17] obtained for $a_1 = 0.6$ and $0 < a_4 \leq 1.07$.

of unbounded orbits are in white, fixed points in gray, periodic in blue, quasi-periodic in green, and chaotic in red. We use $|LE| < 0.0001$ as the criterion for quasi-periodic orbits with 10^6 iterations for each point. Thus the two chaotic systems go via different and distinguishable routes to chaos. More generally, due to the smoothness of the quadratic map of the plane given in (1), if a bifurcation occurs, then it is one of the generic types, namely, period-doubling, saddle-node, or Hopf.

8.1 Some observed catastrophic solutions in the dynamics of the map

In this subsection, we investigate some important dynamical behaviors observed for some special cases of the map (1), i.e., we discuss the occurrence of some isolated islands that make "breaks" in the dynamics of the map where such maps pass between two bounded states through unbounded orbits. Such catastrophes render the system unsuitable for many potential applications. This phenomena is not observed for the well known simple 2-D quadratic maps [2-9-17] whose dynamics seem to be a single island of bounded orbits in a "sea" of unbounded orbits. These islands contain all the bounded state orbits of the corresponding map. So, for $a_1 = -0.2$, $a_4 = a_5 = b_3 = b_4 = b_5 = 0$, $b_0 = 0.2$, and $b_2 = 0.2$, i.e., a map of the type

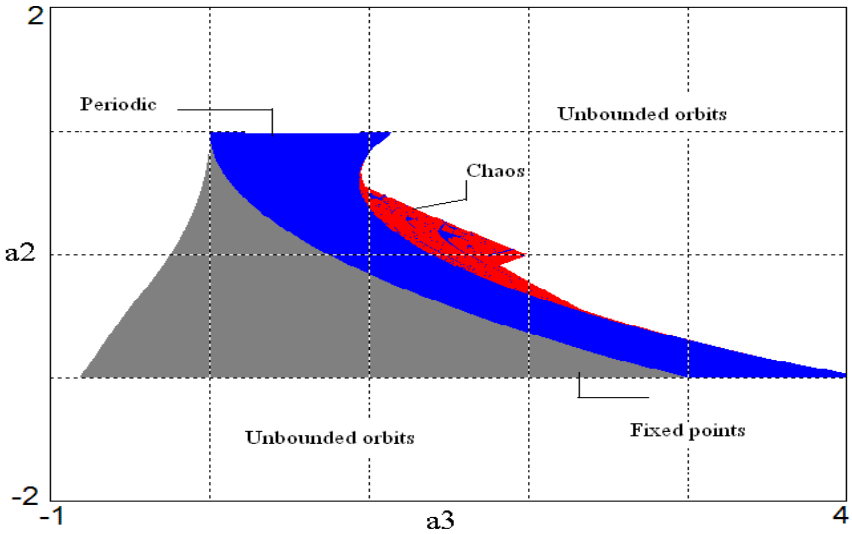

Figure 11: Regions of dynamical behaviors in the a_3-a_2 plane for the Hénon map [2] obtained for $a_1 = a_4 = a_5 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$. [16]

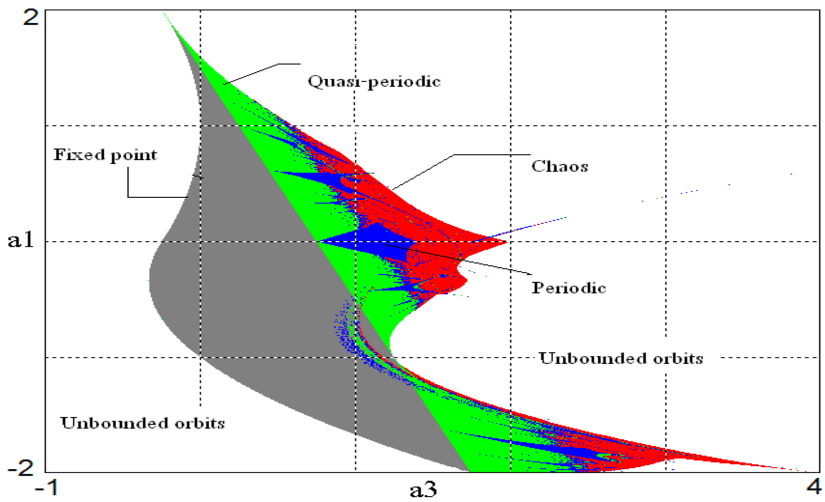

Figure 12: Regions of dynamical behaviors in the a_3-a_1 plane for the map given in [17] obtained for $a_2 = a_4 = a_5 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$.

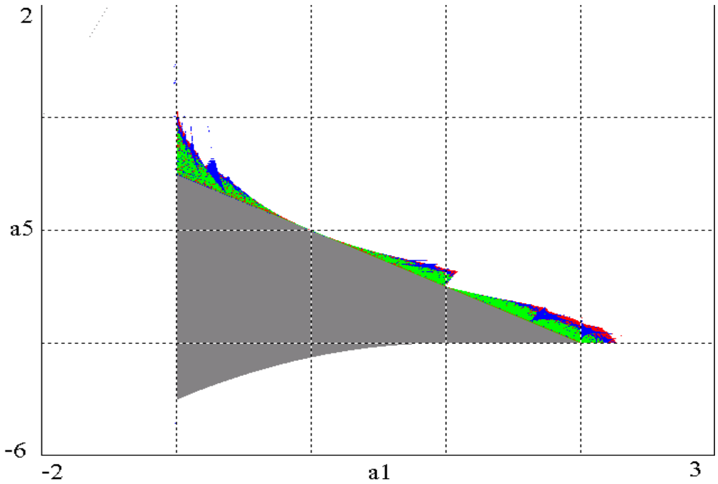

Figure 13: Regions of dynamical behaviors in the a_1-a_5 plane for the map given in [18] for $a_2 = a_3 = a_4 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$.

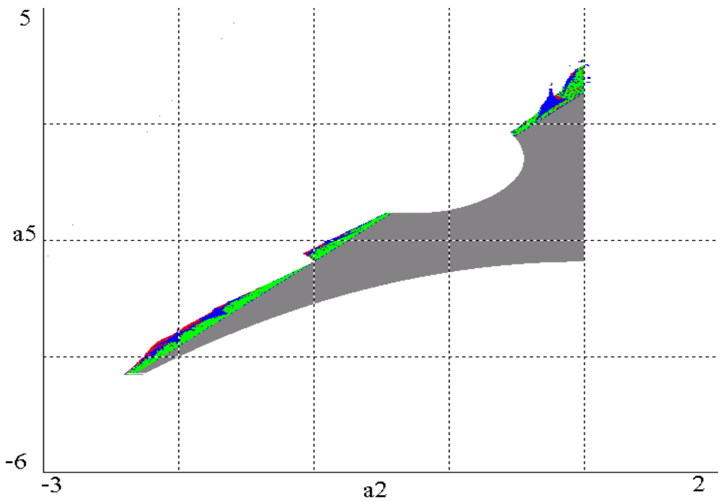

Figure 14: Regions of dynamical behaviors in the a_2-a_5 plane for the map given in [18] for $a_1 = a_3 = a_4 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$.

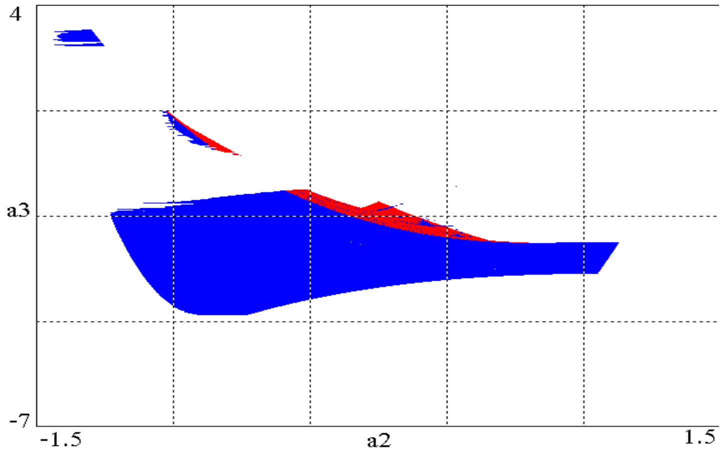

Figure 15: Regions of dynamical behaviors in the a_2-a_3 plane for the map $(1-1)$ for $a_1 = -0.2$, $a_4 = a_5 = b_3 = b_4 = b_5 = 0, b_0 = 0.2$, and $b_2 = 0.2$.

$(1-1)$ with $|a_2| < 1.5$ and $-7 \leq a_3 \leq 4$, we observe some islands of periodic and chaotic orbits with a period-doubling route to chaos surrounded by unbounded orbits, or maybe these unbounded orbits are large chaotic orbits obtained via a period-doubling route to chaos. This phenomenon can be seen in Fig. 15.

On the other hand, and for $a_4 = -1.8$ and $a_2 = a_5 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$, i.e., a map of the type $(2-1)$ with $-3 \leq a_1 \leq 2$ and $-5 \leq a_3 \leq 2$, we observe some islands of chaotic orbits without any route to chaos, surrounded by unbounded orbits, or maybe these unbounded orbits are large chaotic orbits obtained via a quasi-periodic route to chaos. These phenomenon can be seen in Fig. 16.

9 Conclusion

Some new fundamental results about the dynamics 2-D quadratic maps were reported, along with an overview of some common issues related to these mappings.

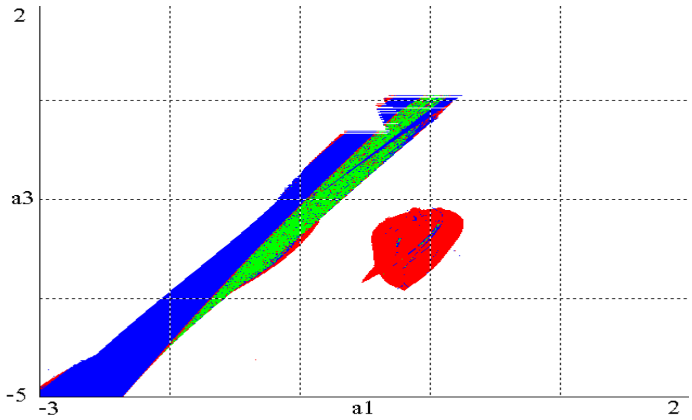

Figure 16: Regions of dynamical behaviors in the a_1-a_3 plane for the map $(2-1)$ for $a_4 = -1.8$ and $a_2 = a_5 = b_0 = b_2 = b_3 = b_4 = b_5 = 0$.

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